PURE MATHEMATICS 1

WORKED SOLUTIONS FOR

CHAPTER 1
OF

Pure Mathematics 1: Coursebook

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Complete solutions for approximately half the exercises in the whole book may be obtained by subscription to the Pure Maths 1 course on www.imago-education.com.

NOTATION AND TERMINOLOGY

Note: not all of these symbols are used in every chapter.

strict inequality	one which doesn't include 'equal to' < and > are strict inequalities
weak inequality	one which includes 'equal to' $\leq$ and $\geq$ are weak inequalities
$\therefore$	*therefore*
$\because$	*because*, or *since*
eqn	*equation*
$\rightarrow$	*substitute into* E.g (1) $\rightarrow$ (2) means 'substitute equation (1) into equation (2)'.
$\Rightarrow$	*implies* E.g. A $\Rightarrow$ B means 'statement A implies statement B'. In other words, if statement A is true, statement B must be true (but not necessarily the other way around).
$\Leftarrow$	*is implied by* E.g. A $\Leftarrow$ B means 'statement A is implied by statement B'. In other words, if statement B is true, statement A must be true (but not necessarily the other way around).
$\Leftrightarrow$	*implies and is implied by* E.g. A $\Leftrightarrow$ B means 'statement A implies and is implied by statement B'. In other words, if statement A is true, statement B must be true AND if statement B is true, statement A must be true.
TBP	*to be proved* (used at beginning of proof)
QED	*which was to be shown* (used at end of proof) (from Latin *quod erat demonstrandum*)

Δ	*change in*
Δ	The discriminant $(= b^2 - 4ac)$ of a quadratic expression in the form $ax^2 + bx + c$.
	Note: the context will indicate which use of Δ is applicable in any particular situation.

EXERCISE 1A

1 b)

$$\begin{aligned} length &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{\big(1 - (-3)\big)^2 + (-1 - 2)^2} \\ &= \sqrt{4^2 + 3^2} \\ &= \sqrt{16 + 9} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

d)

$$\begin{aligned} length &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{\big(-7 - (-3)\big)^2 + \big(3 - (-3)\big)^2} \\ &= \sqrt{(-4)^2 + 6^2} \\ &= \sqrt{16 + 36} \\ &= \sqrt{52} \\ &= \sqrt{4 \times 13} = \sqrt{4}\sqrt{13} = 2\sqrt{13} \end{aligned}$$

f)

$$\begin{aligned} length &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{\big((a - 1) - (a + 1)\big)^2 + \big((2a - 1) - (2a + 3)\big)^2} \\ &= \sqrt{(-2)^2 + (-4)^2} \\ &= \sqrt{4 + 16} = \sqrt{20} \end{aligned}$$

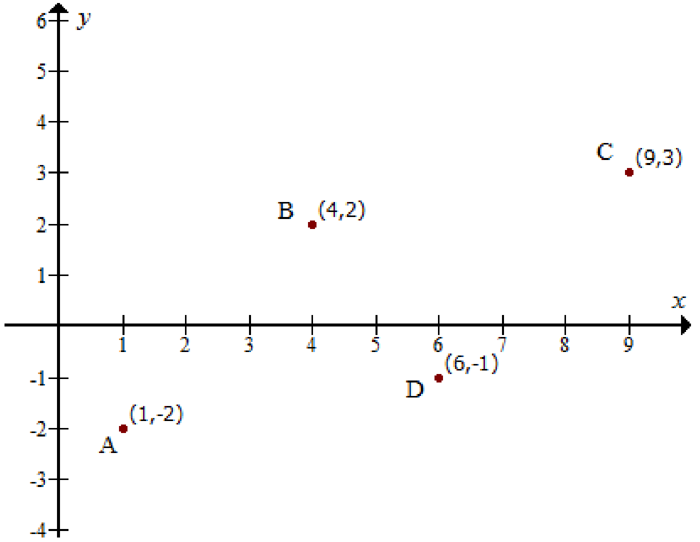
h)

$$\begin{aligned} length &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3a - 12a)^2 + (5b - 5b)^2} \\ &= \sqrt{(-9a)^2 + 0^2} \\ &= \sqrt{81a^2} \\ &= 9a \end{aligned}$$

j)

$$\begin{aligned} length &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{\big((p - 3q) - (p + 4q)\big)^2 + (p - (p - q))^2} \\ &= \sqrt{(-7q)^2 + q^2} \\ &= \sqrt{50q^2} \\ &= q\sqrt{50} \end{aligned}$$

2) Plot the points on the co-ordinate plane:



Now show that a pair of opposite sides is parallel and equal in length.

Take AB, formed by (1,-2) and (4,2):

$$gradient = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2-(-2)}{4-1}$$
$$= \frac{4}{3}$$

$$length = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$
$$= \sqrt{(4-1)^2 + \left(2-(-2)\right)^2}$$
$$= \sqrt{3^2+4^2}$$
$$= 5$$

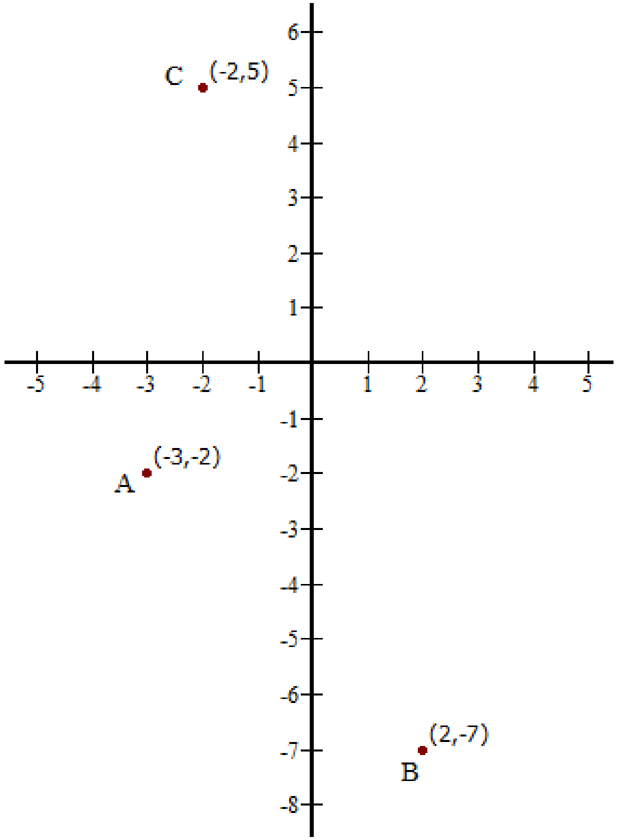
Now take DC, formed by (6,-1) and (9,3):

$$gradient = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{3-(-1)}{9-6}$$
$$= \frac{4}{3}$$

$$length = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$
$$= \sqrt{(9-6)^2 + \left(3-(-1)\right)^2}$$
$$= \sqrt{3^2+4^2} = 5$$

Therefore AB and DC are parallel and equal in length, so the four points form a parallelogram.

3) Plot the three points:



From the diagram it appears most likely that sides AC and AB will be equal. Therefore calculate their lengths:

$$length\ AB = \sqrt{(2-(-3))^2 + (-7-(-2))^2}$$

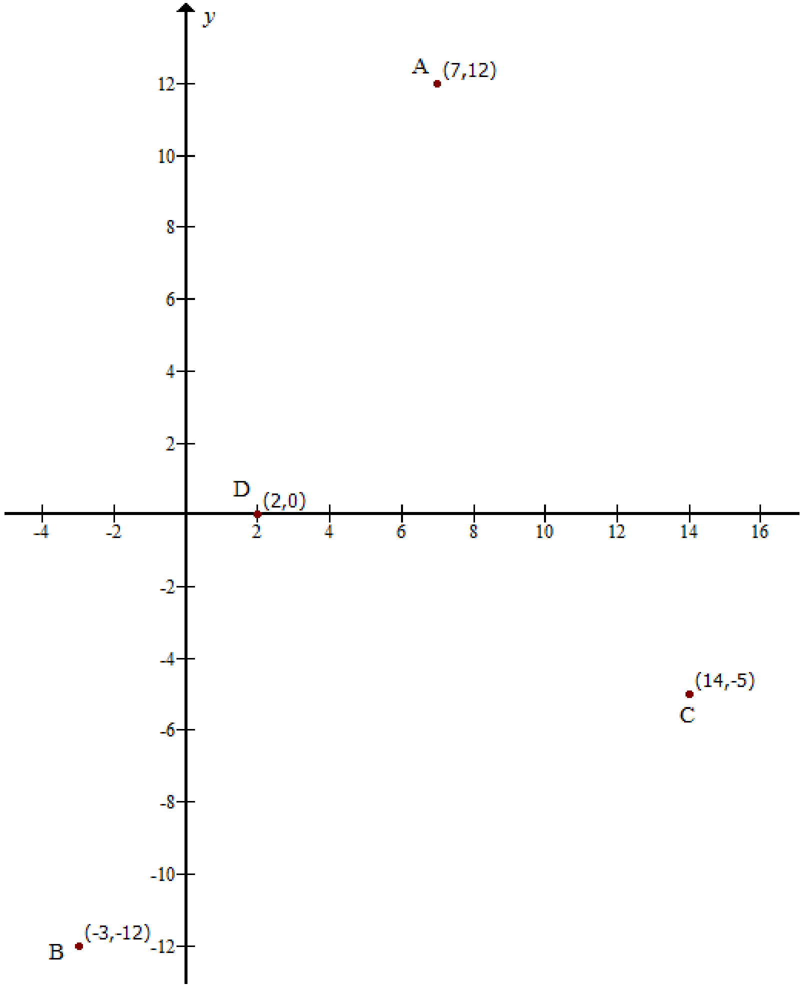
$$= \sqrt{5^2 + 5^2} = \sqrt{50}$$

$$length\ AC = \sqrt{(-2-(-3))^2 + (5-(-2))^2}$$

$$= \sqrt{1^2 + 7^2} = \sqrt{50}$$

Therefore AB = AC, which means that triangle ABC is isosceles.

4) Plot the points and the centre of the circle:



To show that A, B & C lie on a circle with centre D, we must prove that AD = BD = CD:

$$AD = \sqrt{(7-2)^2 + (12-0)^2}$$
$$= \sqrt{5^2 + 12^2} = 13$$

$$BD = \sqrt{\left(2-(-3)\right)^2 + \left(0-(-12)\right)^2}$$
$$= \sqrt{5^2 + 12^2} = 13$$

$$CD = \sqrt{(14-2)^2 + (-5-0)^2}$$
$$= \sqrt{12^2 + 5^2} = 13$$

Therefore AD = BD = CD, which means that A, B, C all lie on a circle with centre D.

(Notice that, when calculating the length of a line segment, one can use either point as the first one. Thus, in calculating CD above, D was used as the first point and C as the second, giving $(14-2)$ and $(-5-0)$ rather than $(2-14)$ and $(0-(-5))$. The result is the same either way because $12^2 = (-12)^2$.)

5 a) Midpoint is $(x, y) = \left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right)$:

$$x = \frac{1}{2}(2+6) = 4$$

$$y = \frac{1}{2}(11+15) = 13$$

Therefore the midpoint is $(4,13)$.

c)

$$x = \frac{1}{2}(-2+1) = -\frac{1}{2}$$

$$y = \frac{1}{2}\left(-3+(-6)\right) = -\frac{9}{2}$$

Therefore midpoint is $\left(-\frac{1}{2}, -\frac{9}{2}\right)$.

e)

$$x = \frac{1}{2}(p+2+3p+4) = \frac{1}{2}(4p+6) = 2p+3$$

$$y = \frac{1}{2}(3p-1+p-5) = \frac{1}{2}(4p-6) = 2p-3$$

Midpoint is $(2p+3, 2p-3)$.

g)

$$x = \frac{1}{2}(p+2q+5p-2q) = \frac{1}{2}(6p) = 3p$$

$$y = \frac{1}{2}\left(2p+13q+(-2p-7q)\right) = \frac{1}{2}(6q) = 3q$$

Midpoint is $(3p, 3q)$.

6) The centre of the circle (let's call it C) is at the midpoint of AB.

$$x_C = \frac{1}{2}(-2+6) = 2$$

$$y_C = \frac{1}{2}(1+5) = 3$$

$$C = (2, 3)$$

7) We use the midpoint formula to set up two equations, which we solve for x_B and y_B:

$$\begin{aligned} x_M &= \frac{1}{2}(x_A + x_B) \\ 5 &= \frac{1}{2}(3 + x_B) \\ 10 &= 3 + x_B \\ x_B &= 7 \end{aligned} \qquad\qquad \begin{aligned} y_M &= \frac{1}{2}(y_A + y_B) \\ 7 &= \frac{1}{2}(4 + y_B) \\ 14 &= 4 + y_B \\ y_B &= 10 \end{aligned}$$

Therefore $B = (7,10)$.

10 a)

$$\begin{aligned} gradient &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{12 - 8}{5 - 3} = \frac{4}{2} = 2 \end{aligned}$$

c)

$$gradient = \frac{-1 - (-3)}{0 - (-4)} = \frac{2}{4} = \frac{1}{2}$$

e)

$$\begin{aligned} gradient &= \frac{(-p - 5) - (p - 3)}{(2p + 4) - (p + 3)} = \frac{-2p - 2}{p + 1} \\ &= \frac{-2(p + 1)}{p + 1} = -2 \end{aligned}$$

g)

$$\begin{aligned} gradient &= \frac{(q - p + 3) - (q + p - 3)}{(p - q + 1) - (p + q - 1)} = \frac{-2p + 6}{-2q + 2} \\ &= \frac{-2(p - 3)}{-2(q - 1)} = \frac{p - 3}{q - 1} \end{aligned}$$

11)

$$gradient_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{6 - 4}{7 - 3} = \frac{2}{4} = \frac{1}{2}$$

$$gradient_{BC} = \frac{y_C - y_B}{x_C - x_B} = \frac{1 - 6}{-3 - 7} = \frac{-5}{-10} = \frac{1}{2}$$

The gradients of AB and BC are equal, which means that the lines AB and BC are parallel. Since they have point B in common, A, B and C must be on the same straight line (i.e. they are collinear).

12)

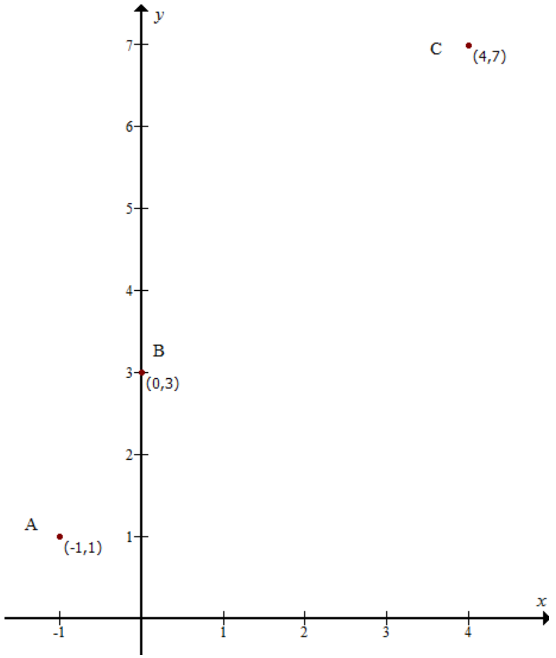
$$gradient_{AP} = \frac{y_P - y_A}{x_P - x_A} = \frac{y - 0}{x - 3} = \frac{y}{x - 3}$$

$$gradient_{PB} = \frac{y_B - y_P}{x_B - x_P} = \frac{6 - y}{5 - x}$$

Since A, P & B are all on the same straight line, $gradient_{AP} = gradient_{PB}$. Therefore

$$\begin{aligned} \frac{y}{x - 3} &= \frac{6 - y}{5 - x} \\ y(5 - x) &= (6 - y)(x - 3) \\ 5y - xy &= 6x - 18 - xy + 3y \\ 2y &= 6x - 18 \\ y &= 3x - 9 \end{aligned}$$

13) Draw the points:



(Note: it looks like the points are almost on the same straight line, rather than forming a triangle. Geometrically, this can be a little confusing, but the algebraic calculations are exactly the same.)

The median AM will join A and M, with M being the midpoint of BC (since BC is the side opposite to A). We must first find the coordinates of M using the midpoint formula:

$$x_M = \frac{1}{2}(x_B + x_C) = \frac{1}{2}(0+4) = 2$$

$$y_M = \frac{1}{2}(y_B + y_C) = \frac{1}{2}(3+7) = 5$$

$$\therefore M = (2,5)$$

Now we can find the length of AM using the distance formula:

$$length_{AM} = \sqrt{(x_M - x_A)^2 + (y_M - y_A)^2}$$
$$= \sqrt{\left(2-(-1)\right)^2 + (5-1)^2}$$
$$= \sqrt{3^2+4^2} = \sqrt{9+16} = 5$$

16) We first find the coordinates of P, Q, R & S using the midpoint formula:

$$x_P = \frac{1}{2}(1+7) = 4 \qquad x_Q = \frac{1}{2}(7+9) = 8$$

$$y_P = \frac{1}{2}(1+3) = 2 \qquad y_Q = \frac{1}{2}\left(3+(-7)\right) = -2$$

$$\therefore P = (4,2) \qquad \therefore Q = (8,-2)$$

$$x_R = \frac{1}{2}\left(9+(-3)\right) = 3 \qquad x_S = \frac{1}{2}(-3+1) = -1$$

$$y_R = \frac{1}{2}\left(-7+(-3)\right) = -5 \qquad y_S = \frac{1}{2}(-3+1) = -1$$

$$\therefore R = (3,-5) \qquad \therefore S = (-1,-1)$$

Now we calculate the gradients of the four sides using the gradient formula with the coordinates of the points just calculated:

$$gradient_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{-2-2}{8-4} = \frac{-4}{4} = -1$$

$$gradient_{QR} = \frac{-5-(-2)}{3-8} = \frac{-3}{-5} = \frac{3}{5}$$

$$gradient_{RS} = \frac{-1-(-5)}{-1-3} = \frac{4}{-4} = -1$$

$$gradient_{SP} = \frac{2-(-1)}{4-(-1)} = \frac{3}{5}$$

Thus, the opposite sides PQ and RS are parallel, and the opposite sides QR and SP are parallel; therefore the quadrilateral PQRS is a parallelogram.

18 a) We use the distance formula:

$$ON = \sqrt{(6-0)^2 + (4-0)^2} = \sqrt{6^2 + 4^2} = \sqrt{60}$$

$$LM = \sqrt{\left(4-(-2)\right)^2 + (7-3)^2} = \sqrt{6^2 + 4^2} = \sqrt{60}$$

Thus $ON = LM$ (qed).

b) We calculate gradients using the gradient formula:

$$gradient_{ON} = \frac{4-0}{6-0} = \frac{2}{3}$$

$$gradient_{LM} = \frac{7-3}{4-(-2)} = \frac{4}{6} = \frac{2}{3}$$

Therefore ON is parallel to LM (since their gradients are equal).

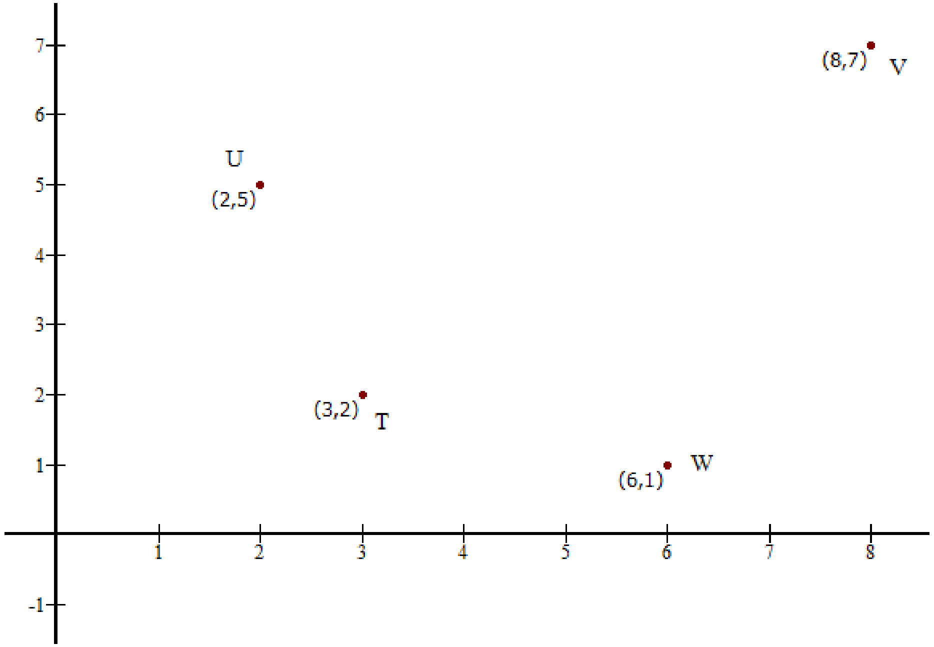
c) We use the distance formula:

$$OM = \sqrt{(4-0)^2 + (7-0)^2} = \sqrt{4^2 + 7^2} = \sqrt{16+49} = \sqrt{65}$$

$$LN = \sqrt{\left(6-(-2)\right)^2 + (4-3)^2} = \sqrt{8^2 + 1^2} = \sqrt{65}$$

d) OLMN is a quadrilateral with two opposite sides (ON and LM) equal and parallel. This means that OLMN is a parallelogram. In addition, the diagonals OM and LN are equal. Therefore OLMN is a **rectangle**.

20) We start by plotting the points:



We calculate the coordinates of M and N using the midpoint formula:

$$x_M = \frac{1}{2}(x_U + x_V) = \frac{1}{2}(2+8) = 5$$

$$y_M = \frac{1}{2}(y_U + y_V) = \frac{1}{2}(5+7) = 6$$

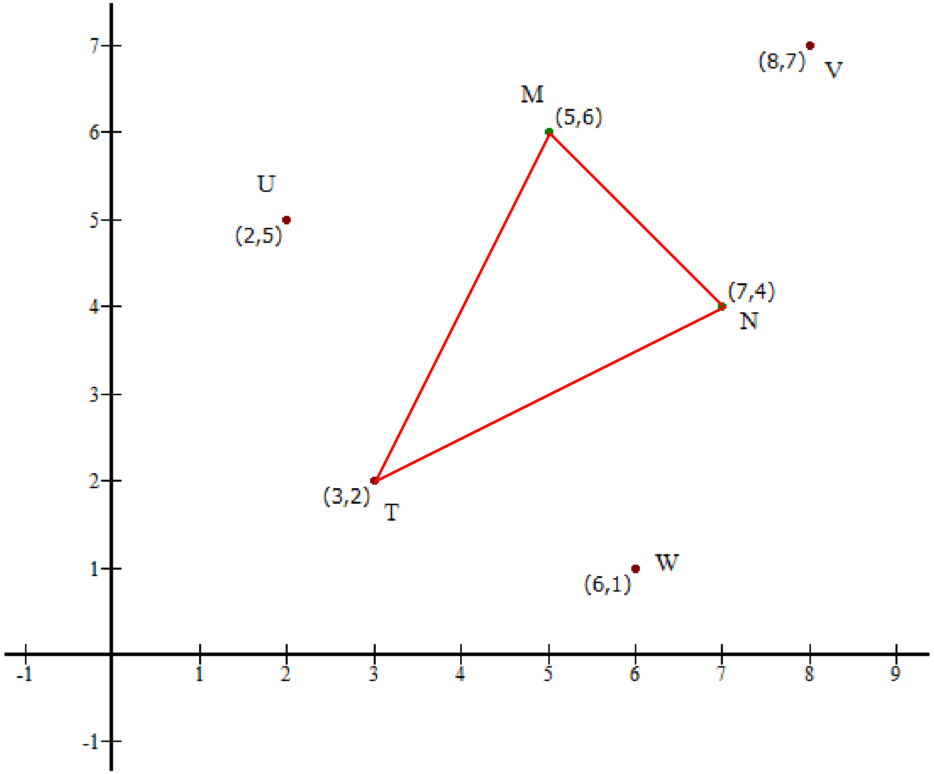
$$\therefore M = (5,6)$$

$$x_N = \frac{1}{2}(x_W + x_V) = \frac{1}{2}(6+8) = 7$$

$$y_N = \frac{1}{2}(y_W + y_V) = \frac{1}{2}(1+7) = 4$$

$$\therefore N = (7,4)$$

To help visualize the points and the triangle TMN, it helps to add the points M and N, and to draw in the triangle:



From the diagram we can see that TM and TN are likely to be the equal sides, therefore we calculate the lengths of these sides using the distance formula:

$$TM = \sqrt{(5-3)^2 + (6-2)^2} = \sqrt{2^2 + 4^2} = \sqrt{20}$$

$$TN = \sqrt{(7-3)^2 + (4-2)^2} = \sqrt{4^2 + 2^2} = \sqrt{20}$$

Therefore TM=TN and triangle TMN is isosceles.

22 a) We start off using the midpoint formula:

$$x_M = \frac{1}{2}(2+10) = 6$$

$$y_M = \frac{1}{2}(1+1) = 1$$

$$\therefore M = (6,1)$$

b) First, we use the distance formula:

$$BG = \sqrt{(6-6)^2 + (4-10)^2} = \sqrt{36} = 6$$

$$GM = \sqrt{(6-6)^2 + (1-4)^2} = \sqrt{9} = 3$$

$$\therefore BG = 2GM$$

To prove that BGM is a straight line, we notice that B, G & M all have the same x-coordinate. Therefore they all lie on the line $x = 6$. Note that we cannot prove this using gradients because a vertical line has an infinite gradient. (Try calculating the gradient of BG, and you will see that you get a division-by-zero error.)

c) We use the midpoint formula:

$$x_N = \frac{1}{2}(6+10) = 8$$

$$y_N = \frac{1}{2}(10+1) = \frac{11}{2}$$

$$\therefore N = (8, \frac{11}{2})$$

d)

$$gradient_{AG} = \frac{4-1}{6-2} = \frac{3}{4}$$

$$gradient_{GN} = \frac{\frac{11}{2}-4}{8-6} = \frac{3}{2}\times\frac{1}{2} = \frac{3}{4}$$

Therefore A, G, N are all on the same straight line.

Now we use the distance formula:

$$AG = \sqrt{(6-2)^2+(4-1)^2} = \sqrt{16+9} = 5$$

$$GN = \sqrt{(8-6)^2+\left(\frac{11}{2}-4\right)^2} = \sqrt{4+\frac{9}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

Therefore $AG = 2GN$.

EXERCISE 1B

1) To check whether a point lies on a line, we must check whether the left and right hand sides of the equation are equal when the x and y values of the point are substituted into the equation of the line.

a) $(1,2)$ on $y = 5x - 3$:

$$LHS = 2$$

$$RHS = 5(1) - 3 = 2$$

Therefore the point $(1,2)$ does lie on the line $y = 5x - 3$.

b) $(3,-2)$ on $y = 3x - 7$:

$$LHS = -2$$

$$RHS = 3(3) - 7 = 9 - 7 = 2$$

Therefore the point $(3,-2)$ does not lie on the line $y = 3x - 7$.

d) $(2,2)$ on $3x^2 + y^2 = 40$:

$$LHS = 3(2)^2 + (2)^2 = 3(4) + 4 = 16$$

$$RHS = 40$$

Therefore the point $(2,2)$ does not lie on the line $3x^2 + y^2 = 40$.

f) $(5p, \frac{5}{p})$ on $y = \frac{5}{x}$:

$$LHS = \frac{5}{p}$$

$$RHS = \frac{5}{5p} = \frac{1}{p}$$

Therefore the point $\left(5p, \frac{5}{p}\right)$ does not lie on the line $y = \frac{5}{x}$.

h) $(t^2, 2t)$ on $y^2 = 4x$:

$$LHS = (2t)^2 = 4t^2$$

$$RHS = 4(t^2) = 4t^2$$

Therefore the point $(t^2, 2t)$ does lie on the line $y^2 = 4x$.

2) In each case, let the equation of the line be $y = mx + c$, where m is the gradient of the line and c is the y intercept.

b)

$$m = -3$$

$$y = -3x + c$$

Substitute $(1, -2)$ into this equation to get c:

$$-2 = -3(1) + c$$

$$c - 3 = -2$$

$$c = 1$$

Therefore the equation is $y = -3x + 1$.

d)

$$y = -\frac{3}{8}x + c$$

$$1 = -\frac{3}{8}(-2) + c$$

$$1 = \frac{6}{8} + c$$

$$8c + 6 = 8$$

$$8c = 2$$

$$c = \frac{1}{4}$$

Therefore the equation of the line is

$$y = -\frac{3}{8}x + \frac{1}{4}$$

which is equivalent to

$$8y = -3x + 2.$$

f) If the gradient is zero (i.e. the line is horizontal), the equation of the line has the form $y = c$. We can take the y-value of any point on the line to get the value of c. Therefore the equation of the line is

$$y = 8$$

h)

$$y = \frac{1}{2}x + c$$

$$0 = \frac{1}{2}(-3) + c$$

$$0 = c - \frac{3}{2}$$

$$c = \frac{3}{2}$$

Therefore the equation of the line is

$$y = \frac{1}{2}x + \frac{3}{2}$$

or

$$2y = x + 3$$

j)

$$y = -\frac{1}{2}x + c$$
$$4 = -\frac{1}{2}(3) + c$$
$$8 = -3 + 2c$$
$$2c = 11$$
$$c = \frac{11}{2}$$

Equation is:

$$y = -\frac{1}{2}x + \frac{11}{2}$$

or

$$2y = -x + 11$$

l)

$$y = 3x + c$$
$$-5 = 3(-2) + c$$
$$-5 = -6 + c$$
$$c = 1$$

Equation is:

$$y = 3x + 1$$

n)

$$y = -x + c$$
$$2 = -(0) + c$$
$$c = 2$$

Equation is:

$$y = -x + 2$$

p)

$$y = -\frac{3}{5}x + c$$
$$0 = -\frac{3}{5}(3) + c$$
$$0 = -\frac{9}{5} + c$$
$$c = \frac{9}{5}$$

Equation is:

$$y = -\frac{3}{5}x + \frac{9}{5}$$
$$5y = -3x + 9$$

r)

$$y = mx + c$$
$$4 = m(0) + c$$
$$c = 4$$

Equation is:

$$y = mx + 4$$

t)

$$y = mx + b$$

(Use b instead of c here to avoid confusion with the value of x, which is given as c.)

$$0 = mc + b$$

$$b = -mc$$

Equation is:

$$y = mx - mc$$

3 b)

$$gradient = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7 - 5}{-2 - 4} = \frac{-12}{-6} = 2$$

Therefore the equation of the line is $y = 2x + c$. To find the value of c, substitute the x and y values of one of the points:

$$5 = 2(4) + c$$

$$c = 5 - 8 = -3$$

$$\therefore y = 2x - 3$$

f)

$$gradient = \frac{20 - (-1)}{-4 - 3} = \frac{21}{-7} = -3$$

Equation of line: $y = -3x + c$. At $(3, -1)$:

$$-1 = -3(3) + c$$

$$c = -1 + 9 = 8$$

$$\therefore y = -3x + 8$$

j)

$$gradient = \frac{-3 - (-1)}{5 - (-2)} = \frac{-2}{7} = -\frac{2}{7}$$

Equation of line is $y = -\frac{2}{7}x + c$. At $(-2, -1)$:

$$-1 = -\frac{2}{7}(-2) + c$$

$$-7 = 4 + 7c$$

$$c = -\frac{11}{7}$$

$$\therefore y = -\frac{2}{7}x - \frac{11}{7}$$

$$7y = -2x - 11$$

$$2x + 7y + 11 = 0$$

n)

$$gradient = \frac{-1 - 4}{-2 - (-5)} = \frac{-5}{3} = -\frac{5}{3}$$

Equation of line is $y = -\frac{5}{3}x + c$. At $(-5, 4)$:

$$4 = -\frac{5}{3}(-5) + c$$

$$12 = 25 + 3c$$

$$3c = -13$$

$$c = -\frac{13}{3}$$

$$\therefore y = -\frac{5}{3}x - \frac{13}{3}$$

$$3y = -5x - 13$$

$$5x + 3y + 13 = 0$$

p)

$$gradient = \frac{q-0}{p-0} = \frac{q}{p}$$

Equation of line is $y = \frac{q}{p}x + c$. At $(0,0)$:

$$0 = c$$

$$\therefore y = \frac{q}{p}x$$

$$or \;\; py = qx$$

$$or \;\; qx - py = 0$$

q)

$$gradient = \frac{q-1-q}{p+3-p} = -\frac{1}{3}$$

Equation of line is $y = -\frac{1}{3}x + c$. At (p,q):

$$q = -\frac{1}{3}p + c$$

$$3q = -p + 3c$$

$$3c = 3q + p$$

$$c = \frac{3q+p}{3}$$

$$\therefore y = -\frac{1}{3}x + \frac{3q+p}{3}$$

$$3y = -x + 3q + p$$

$$x + 3y - p - 3q = 0$$

r) Notice that both points have the same x-coordinate, p. This means that both points lie on the vertical line, $x = p$. So, the equation of the line joining these points is $x = p$. This question cannot be answered in the same way as above, since the gradient of a vertical line is infinite.

s)

$$gradient = \frac{(q+2)-q}{(p+2)-p}$$
$$= 1$$

Equation of line is $y = x + c$. At (p,q):

$$q = p + c$$
$$c = q - p$$

$$\therefore y = x + q - p$$

$$or \quad x - y + q - p = 0$$

t)

$$gradient = \frac{q-0}{0-p}$$
$$= -\frac{q}{p}$$

Equation of line is $y = -\frac{q}{p}x + c$. At $(0, q)$:

$$q = -\frac{q}{p}(0) + c$$
$$c = q$$
$$\therefore y = -\frac{q}{p}x + q$$
$$or \quad qx + py - pq = 0$$

4) In each case write the given equation in the form $y = mx + c$ and then read off the gradient (it is the coefficient of x).

a)

$$2x + y = 7$$
$$y = -2x + 7$$

From the form of the equation it is now easy to see that the gradient is -2.

c)

$$5x + 2y = -3$$
$$2y = -5x - 3$$
$$y = -\frac{5}{2}x - \frac{3}{2}$$

So, the gradient is $-\frac{5}{2}$.

f) The line $5x = 7$ is a vertical line at $x = \frac{7}{5}$, so the gradient is infinite or undefined.

h)

$$y = 3(x + 4)$$
$$y = 3x + 12$$

Gradient is 3.

k)

$$y = m(x - d)$$
$$y = mx - md$$

Gradient is m.

l)

$$px + qy = pq$$
$$qy = -px + pq$$
$$y = -\frac{p}{q}x + p$$

Gradient is $-\frac{p}{q}$.

5) Since the line must be parallel to the line $y = \frac{1}{2}x - 3$, the gradient is $\frac{1}{2}$. So the equation of the line is $y = \frac{1}{2}x + c$, and we find the value of c by substituting for x and y at the given point $(-2,1)$:

$$1 = \frac{1}{2}(-2) + c$$

$$c = 1 + 1$$

$$= 2$$

So the equation of the line is $y = \frac{1}{2}x + 2$.

6) First, we find the gradient. The line is parallel to $y + 2x = 7$, which can be written as $y = -2x + 7$. So the gradient is -2.The equation of the line is $y = -2x + c$. At $(4, -3)$:

$$-3 = -2(4) + c$$

$$c = -3 + 8$$

$$= 5$$

The equation of the line is $y = -2x + 5$.

7) Here the gradient is the same as for the line joining the points $(3, -1)$ and $(-5,2)$:

$$gradient = \frac{2 - (-1)}{(-5) - 3}$$

$$= -\frac{3}{8}$$

The equation of the line is $y = -\frac{3}{8}x + c$. At $(1,2)$:

$$2 = -\frac{3}{8}(1) + c$$

$$c = 2 + \frac{3}{8}$$

$$= \frac{19}{8}$$

The equation of the line is $y = -\frac{3}{8}x + \frac{19}{8}$.

8) Here the gradient is the same as for the line joining the points $(-3,2)$ and $(2, -3)$:

$$gradient = \frac{2 - (-3)}{(-3) - 2}$$

$$= -1$$

The equation of the line is $y = -x + c$. At $(3,9)$:

$$9 = -1(3) + c$$

$$c = 9 + 3 = 12$$

The equation of the line is $y = -x + 12$.

9) A line parallel to the x-axis is a horizontal line with a gradient of zero. Thus, the equation of the line that passes through $(1,7)$ and is parallel to the x-axis is $y = 7$.

10) Since the line must be parallel to $y = mx + c$, the gradient of the line is m. The equation of the line is $y = mx + b$ (using b to avoid confusion). At $(d, 0)$:

$$0 = md + b$$

$$b = -md$$

The equation of the line is $y = mx - md$.

11) The point of intersection of two lines is the point (x, y) that satisfies both equations.

b) The simplest approach here is to equate the right hand sides of each equation and solve for x:

$$3x + 1 = 4x - 1$$
$$x = 2$$

Now substitute this value of x into either of the given equations:

$$y = 3(2) + 1$$
$$y = 7$$

Check this answer by substituting these values for x and y back into the second equation (the one that was not used to solve for y) and check that both sides are equal:

$$RHS = 4(2) - 1 = 7$$
$$LHS = 7$$
$$\therefore LHS = RHS$$

So the point of intersection is $(2,7)$.

d) Use the same approach as above.

$$3x + 8 = -2x - 7$$
$$5x = -15$$
$$x = -3$$

Substitute for x in the first equation:

$$y = 3(-3) + 8$$
$$y = -1$$

Check:

$$RHS = -2(-3) - 7 = -1$$
$$LHS = -1$$
$$\therefore LHS = RHS$$

So the point of intersection is $(-3, -1)$.

f) Here it is simpler to multiply the first equation by 5 and the second equation by -2 and then to add the equations to eliminate x:

$$10x + 35y = 235$$
$$-10x - 8y = -100$$

Adding these equations and solving for y:

$$27y = 135$$
$$y = 5$$

Substituting for y in the first equation and solving for x:

$$2x + 7(5) = 47$$
$$2x = 12$$
$$x = 6$$

Check by substituting these values for x and y into the second equation:

$$LHS = 5(6) + 4(5) = 30 + 20 = 50 = RHS$$

So the point of intersection is $(6,5)$.

g) Multiply the first equation by -3 and add the two equations to eliminate x:

$$-6x - 9y = -21 \qquad (1)$$

$$6x + 9y = 11 \qquad (2)$$

(1) + (2):

$$0 = -10$$

This is obviously not true. These equations cannot be solved simultaneously, so we conclude that these two lines do not intersect. This can be understood geometrically by writing the equations in the form $y = mx + c$:

$$y = -\frac{2}{3}x + \frac{7}{3}$$
$$y = -\frac{2}{3}x + \frac{11}{9}$$

These lines have the same gradient so they are parallel lines and never intersect. (The y-intercepts are different so they are not collinear. In the case of collinear lines, there are infinitely many points which will simultaneously satisfy both the equations.)

h) Solve the first equation for y:

$$y = -3x + 5$$

Then substitute this expression for y into the second equation and solve for x:

$$x + 3(-3x + 5) = -1$$
$$x - 9x + 15 = -1$$
$$-8x = -16$$
$$x = 2$$

Then solve for y by substituting this value of x into the first expression for y:

$$y = -3(2) + 5$$
$$y = -1$$

Check by substituting these values for x and y into the second equation:

$$LHS = 2 + 3(-1) = -1 = RHS$$

The lines intersect at $(2, -1)$.

i) The first equation is already written as an expression for y, so we just substitute this expression into the second equation:

$$4x - 2(2x + 3) = -6$$
$$4x - 4x - 6 = -6$$
$$-6 = -6$$

This is always true. Since the x's cancel out in the second line, any value of x will simultaneously satisfy these equations. Thus, any value of y will also simultaneously satisfy both these equations. There are thus infinitely many solutions.

This can be seen geometrically as well. Rewriting the equations in the form $y = mx + c$ gives:

$$y = 2x + 3$$
$$y = 2x + 3$$

So the given equations actually represent the same line.

j) The second equation here is already written as an expression for y, so we just substitute that expression into the first equation:

$$ax + b(2ax) = c$$
$$(a + 2ab)x = c$$
$$x = \frac{c}{a + 2ab}$$

To find y we substitute this expression back into the second equation:

$$y = 2a(\frac{c}{a + 2ab})$$
$$y = \frac{2ac}{a + 2ab}$$
$$y = \frac{2c}{1 + 2b}$$

To check we substitute these expressions for x and y into the first equation:

$$LHS = a\left(\frac{c}{a + 2ab}\right) + b\left(\frac{2c}{1 + 2b}\right)$$
$$= \frac{ac + 2abc}{a + 2ab} = \left(\frac{a + 2ab}{a + 2ab}\right)c$$
$$= c = RHS$$

The lines intersect at $(\frac{c}{a+2ab}, \frac{2ac}{a+2ab})$.

k) Add the two equations to eliminate x:

$$2y = c + d$$
$$y = \frac{c + d}{2}$$

Substitute this expression for y into the first equation:

$$\frac{c + d}{2} = mx + c$$
$$mx = \frac{c + d}{2} - c$$
$$x = \frac{c + d - 2c}{2m}$$
$$x = \frac{d - c}{2m}$$

Check by substituting for x and y into the second equation:

$$RHS = -m\left(\frac{d - c}{2m}\right) + d$$
$$= \frac{c - d}{2} + \frac{2d}{2}$$
$$= \frac{c + d}{2} = LHS$$

So the lines intersect at $(\frac{d-c}{2m}, \frac{c+d}{2})$.

l) Substitute the expression for y from the second equation into the first equation:

$$ax - bx = 1$$
$$(a - b)x = 1$$
$$x = \frac{1}{a - b}$$

Substitute this back into the second equation and to obtain y:

$$y = \frac{1}{a - b}$$

Check, using the first equation:

$$LHS = a\left(\frac{1}{a-b}\right) - b\left(\frac{1}{a-b}\right) = \frac{a-b}{a-b} = 1 = RHS$$

So the lines intersect at $(\frac{1}{a-b}, \frac{1}{a-b})$.

12) Since $P\ (p.q)$ satisfies the equation $y = mx + c$, we have

$$q = mp + c. \qquad (1)$$

The point $Q\ (r, s)$ is any other point on the line $y = mx + c$ so it also satisfies the equation $y = mx + c$ and we can write

$$s = mr + c. \qquad (2)$$

Now the gradient of the line joining the points P and Q is given as usual by:

$$gradient = \frac{s-q}{r-p}.$$

Substituting for q and s from eqns (1) and (2) gives:

$$gradient = \frac{mr + c - (mp + c)}{r-p}$$

$$gradient = \frac{mr - mp}{r-p}$$

$$gradient = \frac{r-p}{r-p}m$$

$$gradient = m$$

Thus, the gradient of the line segment joining the points P and Q is m. (Since P and Q are arbitrary points on the line, this shows that if you have an equation of a line in the form $y = mx + c$, then the coefficient of x is the gradient of that line, a fact that we used in exercise 4.)

13) If $a = b = c = 0$, then any value of x and y will satisfy the equation $ax + by + c = 0$. This equation then represents the entire xy-plane rather than a straight line.

EXERCISE 1C

1) The gradient of a line that is perpendicular to a given line, is the negative of the reciprocal of the gradient of the given line.

c) $-\frac{4}{3}$

d) $\frac{6}{5}$

g) m

h) $-\frac{1}{m}$

k) $\frac{1}{m}$

l) $-\frac{1}{\frac{a}{b-c}} = -\frac{b-c}{a} = \frac{c-b}{a}$

2 b) The gradient of the given line is $-\frac{1}{2}$, so the gradient of the line is 2. So the equation of the line is $y = 2x + c$. At $(-3,1)$:

$$1 = 2(-3) + c$$

$$c = 7$$

$$\therefore y = 2x + 7$$

d) The line $y = 2\frac{1}{2} = \frac{5}{2}$ is a horizontal line. So, any vertical line is perpendicular to this line. So the equation of the line is $x = 7$.

f) Rewrite the given line in the form $y = mx + c$:

$$3x - 5y = 8$$

$$5y = 3x - 8$$

$$y = \frac{3}{5}x - \frac{8}{5}$$

So the gradient is $-\frac{5}{3}$. The equation of the line is $y = -\frac{5}{3}x + c$. At $(4,3)$:

$$3 = -\frac{5}{3}(4) + c$$

$$c = 3 + \frac{20}{3}$$

$$c = \frac{29}{3}$$

$$\therefore \ y = -\frac{5}{3}x + \frac{29}{3}$$

$$or \ \ 3y + 5x = 29$$

h) The gradient is $-\frac{1}{2}$. The equation of the line is $y = -\frac{1}{2}x + c$. At $(0,3)$:

$$3 = -\frac{1}{2}(0) + c$$

$$c = 3$$

$$\therefore \ y = -\frac{1}{2}x + 3$$

$$or \ \ 2y + x = 6$$

j) The gradient is $-\frac{1}{m}$. The equation of the line is $y = -\frac{1}{m}x + d$. At (a,b):

$$b = -\frac{1}{m}a + d$$

$$d = b + \frac{a}{m}$$

$$d = \frac{mb + a}{m}$$

l)

$$ax + by = c$$

$$by = -ax + c$$

$$y = -\frac{a}{b}x + \frac{c}{b}$$

So the gradient of the line is $\frac{b}{a}$. The equation of the line is $y = \frac{b}{a}x + d$. At $(-1, -2)$:

$$-2 = \frac{b}{a}(-1) + d$$

$$d = \frac{b}{a} - 2 = \frac{b - 2a}{a}$$

$$\therefore \quad y = \frac{b}{a}x + \frac{b - 2a}{a}$$

$$or \;\; bx - ay = 2a - b$$

3) The gradient of the line is $-\frac{1}{3}$. The equation of the line is $y = 3x + c$. At $(-2,5)$:

$$5 = -\frac{1}{3}(-2) + c$$

$$c = 5 - \frac{2}{3}$$

$$c = \frac{13}{3}$$

So the equation of the line is $y = -\frac{1}{3}x + \frac{13}{3}$.

To find the point of intersection we equate right hand sides of the two equations:

$$3x + 1 = -\frac{1}{3}x + \frac{13}{3}$$

$$9x + 3 = -x + 13$$

$$10x = 10$$

$$x = 1$$

Using this, we can solve for y:

$$y = 3(1) + 1$$

$$y = 4$$

So the two lines intersect at the point $(1,4)$.

4) Rewrite the given equation in the form $y = mx + c$:

$$2x - 3y = 12$$

$$3y = 2x - 12$$

$$y = \frac{2}{3}x - 4 \qquad (1)$$

So the gradient of the line will be $-\frac{3}{2}$. The equation is $y = -\frac{3}{2}x + c$. At $(1,1)$:

$$1 = -\frac{3}{2}(1) + c$$

$$c = 1 + \frac{3}{2} = \frac{5}{2}$$

$$\therefore \quad y = -\frac{3}{2}x + \frac{5}{2} \qquad (2)$$

To find the point of intersection, substitute eqn (1) into eqn (2):

$$\frac{2}{3}x - 4 = -\frac{3}{2}x + \frac{5}{2}$$

$$4x - 24 = -9x + 15$$

$$13x = 39$$

$$x = 3$$

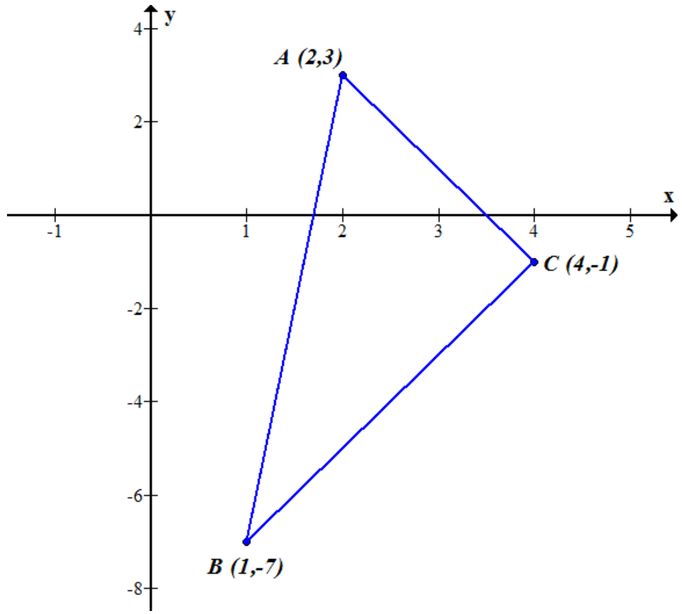
Substitute this result into eqn (1) to solve for y:

$$y = \frac{2}{3}(3) - 4$$

$$y = -2$$

So the lines intersect at $(3, -2)$.

5) First plot the points of the triangle in a coordinate plane:



Now we are looking for the line that is perpendicular to the line segment BC and passes through the point $A\ (2,3)$. So the first step is to find the gradient of the line segment BC:

$$gradient\ of\ BC = \frac{-1-(-7)}{4-1} = \frac{6}{3} = 2$$

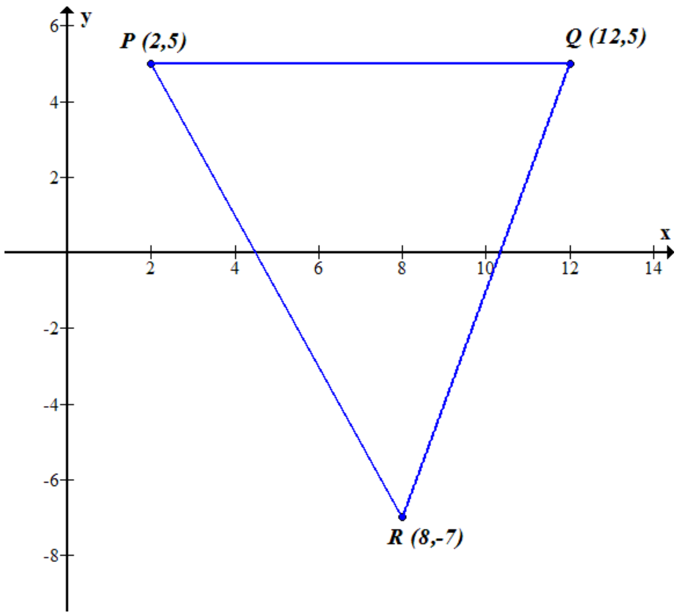
So the gradient of the line we are looking for is $-\frac{1}{2}$. The equation of the line is $y = -\frac{1}{2}x + c$. At $(2,3)$:

$$3 = -\frac{1}{2}(2) + c$$

$$c = 4$$

So the equation of the altitude through the vertex A is $y = -\frac{1}{2}x + 4$.

6) Plot the triangle:



a) The line segment PQ is opposite to the vertex R. Since the line segment PQ is a horizontal line, the altitude through R is a vertical line passing through $(8,-7)$. The equation of this line is $x = 8$.

The altitude through the vertex Q is perpendicular to the line segment PR.

$$gradient\ of\ PR = \frac{5-(-7)}{2-8} = \frac{12}{-6} = -2$$

So the gradient of the altitude through Q is $\frac{1}{2}$. The equation of the line is $y = \frac{1}{2}x + c$. At Q $(12,5)$:

$$5 = \frac{1}{2}(12) + c$$

$$c = -1$$

So the equation of the altitude through Q is $y = \frac{1}{2}x - 1$.

b) Find the point of intersection:

$$x = 8 \qquad (1)$$

$$y = \frac{1}{2}x - 1 \qquad (2)$$

(1) → (2):

$$y = \frac{1}{2}(8) - 1 = 3$$

So these two altitudes intersect at the point $(8,3)$.

c) To show that the altitude through P also passes through the point $(8,3)$, we first find the equation of this altitude and then show that the point $(8,3)$ satisfies that equation.

$$gradient\ of\ QR = \frac{5-(-7)}{12-8} = 3$$

So the gradient of the altitude is $-\frac{1}{3}$ and the equation of the line is $y = -\frac{1}{3}x + c$. At $(2,5)$:

$$5 = -\frac{1}{3}(2) + c$$

$$c = 5 + \frac{2}{3} = \frac{17}{3}$$

So the equation of the altitude through P is $y = -\frac{1}{3}x + \frac{17}{3}$.

Now we show that the point $(8,3)$ satisfies this equation by finding the value of y if we substitute $x = 8$ into the equation:

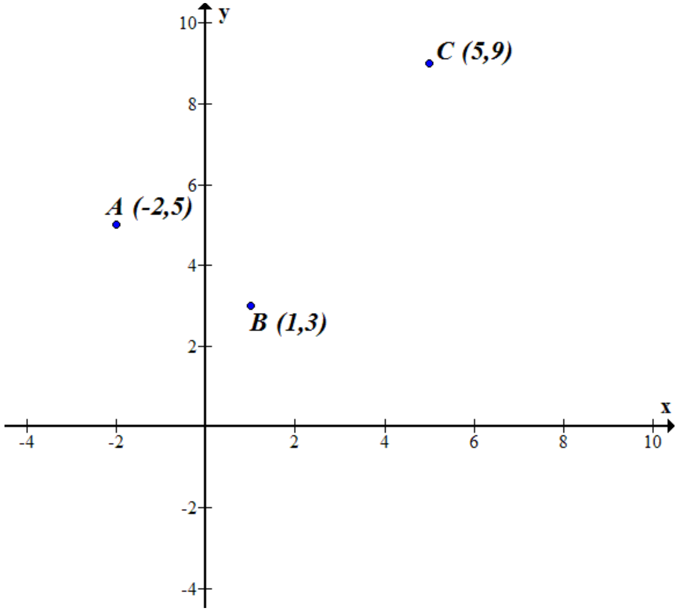
$$y = -\frac{1}{3}(8) + \frac{17}{3}$$

$$y = \frac{9}{3} = 3$$

So the altitude through P also passes through $(8,3)$.

MISCELLANEOUS EXERCISE 1

1) In order to show that a triangle is right-angled, we have to show that two of the line segments joining the vertices are perpendicular. First we plot and label the vertices of the triangle:



From the graph it looks most likely that the line segments AB and BC will be perpendicular. To show this we calculate the gradients of each line segment:

$$gradient\ of\ AB = \frac{3-5}{1-(-2)} = -\frac{2}{3}$$

$$gradient\ of\ BC = \frac{9-3}{5-1} = \frac{6}{4} = \frac{3}{2}$$

We see that the gradient of AB is the negative of the reciprocal of the gradient of BC. So these two line segments are perpendicular.

2) Find the point of intersection:

$$2x + y = 3 \qquad (1)$$

$$3x + 5y - 1 = 0 \qquad (2)$$

Rearrange eqn (1):

$$y = -2x + 3 \qquad (3)$$

(3) → (2):

$$3x + 5(-2x + 3) - 1 = 0$$

$$3x - 10x + 15 - 1 = 0$$

$$7x = 14$$

$$x = 2 \qquad (4)$$

(4) → (3):

$$y = -2(2) + 3$$

$$y = -1 \qquad (5)$$

So the lines intersect at $(2, -1)$.

3a) To show that the angle ACB is a right angle, we have to show that the line segments AC and CB are perpendicular:

$$gradient\ of\ AC = \frac{8-3}{0-(-1)} = 5$$

$$gradient\ of\ CB = \frac{8-7}{0-5} = -\frac{1}{5}$$

So the gradient of AC is the negative of the reciprocal of the gradient of CB. So these line segments are perpendicular to each other and it follows that the angle between them (angle ACB) is a right angle.

b) First we need to find the equation of the line that is parallel to AC and passes through B. Since it is parallel to AC it has the same gradient as AC which is 5 (from part (a)). So the equation of the line is $y = 5x + c$. At B $(5,7)$:

$$7 = 5(5) + c$$

$$c = 7 - 25 = -18$$

So the equation of the line is $y = 5x - 18$.

To find the point where this line crosses the x-axis, we set y to zero and solve for x:

$$0 = 5x - 18$$

$$5x = 18$$

$$x = 3.6$$

So this line crosses the x-axis at the point $(3.6,0)$.

4a) The diagonals of a square have the same length and bisect each other at $90°$. So the diagonal BD is perpendicular to AC and passes through the midpoint of AC.

$$midpoint\ of\ AC = \left(\frac{1}{2}(7+1), \frac{1}{2}(2+4)\right)$$

$$= (4,3)$$

$$gradient\ of\ AC = \frac{4-2}{1-7} = \frac{2}{-6} = -\frac{1}{3}$$

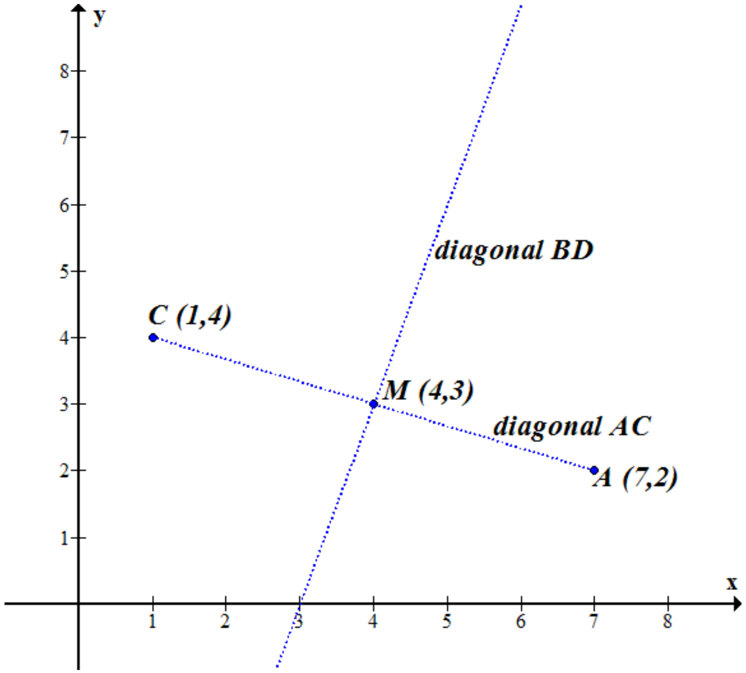
So the gradient of the diagonal BD is 3. The equation of the diagonal is $y = 3x + c$. At $(4,3)$:

$$3 = 3(4) + c$$

$$c = -9$$

So the equation of the diagonal BD is $y = 3x - 9$.

b) Since the diagonals bisect each other and have the same length, the distance from the midpoint of AC to either point B or D is half the distance from point A to point C. Both points B and D also lie on the diagonal BD. First we plot all the information we know:



$$distance\ from\ A\ to\ C = \sqrt{(7-1)^2 + (2-4)^2}$$
$$= \sqrt{40}$$
$$= 2\sqrt{10}$$

So the distance from the midpoint $(4,3)$ to either point B or D is $\sqrt{10}$. Suppose we have a point (x, y) that lies on the diagonal BD and is a distance $\sqrt{10}$ from the midpoint. Then it must satisfy the following equations:

$$\sqrt{(x-4)^2 + (y-3)^2} = \sqrt{10} \qquad (1)$$
$$y = 3x - 9 \qquad (2)$$

Starting from eqn (1):

$$(x-4)^2 + (y-3)^2 = 10 \qquad (3)$$

(2) → (3):

$$(x-4)^2 + (3x-9-3)^2 = 10$$
$$(x-4)^2 + (3x-12)^2 = 10$$
$$x^2 - 8x + 16 + 9x^2 - 72x + 144 = 10$$
$$10x^2 - 80x + 150 = 0$$
$$x^2 - 8x + 15 = 0$$
$$(x-5)(x-3) = 0$$

$$x = 5\ or\ x = 3$$

This gives two values of x that satisfy equations (1) and (2) Thus there are two points that will satisfy the conditions listed above. This is what we were expecting since B and D satisfy those conditions. So we can find the corresponding y values:

At $x = 5$:

$$y = 3(5) - 9 = 6$$

At $x = 3$:

$$y = 3(3) - 9 = 0$$

So the coordinates of B are $(3,0)$ and the coordinates of D are $(5,6)$.

[Note: The answer at the back of the text book is incorrect.]

6a) Rearrange the equation of l_1:

$$3x + 4y = 16$$

$$y = -\frac{3}{4}x + 4$$

So the gradient of l_2 is $\frac{4}{3}$ and its equation is $y = \frac{4}{3}x + c$. At $(7,5)$:

$$5 = \frac{4}{3}(7) + c$$

$$c = 5 - \frac{28}{3}$$

$$c = -\frac{13}{3}$$

So the equation for l_2 is $y = \frac{4}{3}x - \frac{13}{3}$.

b)

$$y = -\frac{3}{4}x + 4 \qquad (1)$$

$$y = \frac{4}{3}x - \frac{13}{3} \qquad (2)$$

(2) → (1):

$$\frac{4}{3}x - \frac{13}{3} = -\frac{3}{4}x + 4$$

$$16x - 52 = -9x + 48$$

$$25x = 100$$

$$x = 4 \qquad (3)$$

(3) → (1):

$$y = -\frac{3}{4}(4) + 4$$

$$y = 1 \qquad (4)$$

So the lines intersect at $(4,1)$.

c) The perpendicular distance of P from the line l_1 is the distance from P to the point of intersection:

$$d = \sqrt{(7-4)^2 + (5-1)^2}$$

$$= \sqrt{9 + 16}$$

$$= 5$$

9) Rearrange the given equation:

$$2x + 7y = 5$$

$$y = -\frac{2}{7}x + \frac{5}{7}$$

So the gradient of the line is $-\frac{2}{7}$. The equation is $y = -\frac{2}{7}x + c$. At $(1,3)$:

$$3 = -\frac{2}{7}(1) + c$$

$$c = 3 + \frac{2}{7} = \frac{23}{7}$$

So the equation of the line is

$$y = -\frac{2}{7}x + \frac{23}{7}$$

$$or\quad 2x + 7y = 23$$

10) First we find the gradient of the line joining $(2, -5)$ and $(-4, 3)$:

$$gradient = \frac{3 - (-5)}{-4 - 2} = \frac{8}{-6} = -\frac{4}{3}$$

So the gradient of the line we are looking for is $\frac{3}{4}$. The line passes through the midpoint of the line joining $(2, -5)$ and $(-4, 3)$:

$$midpoint = \left(\frac{1}{2}(2 + (-4)), \frac{1}{2}(-5 + 3)\right)$$
$$= (-1, -1)$$

The equation of the line is $y = \frac{3}{4}x + c$. At $(-1, -1)$:

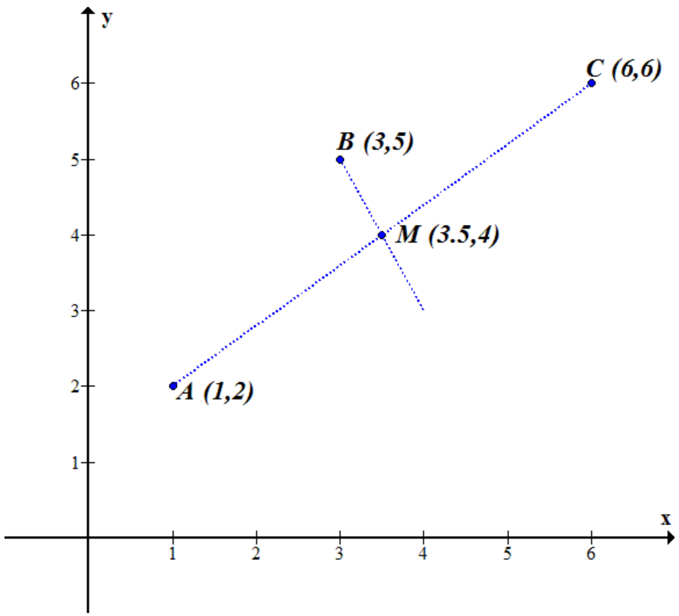
$$-1 = \frac{3}{4}(-1) + c$$
$$c = -1 + \frac{3}{4} = -\frac{1}{4}$$

The equation of the line is $y = \frac{3}{4}x - \frac{1}{4}$.

11)

$$midpoint\ of\ AC = \left(\frac{1}{2}(1 + 6), \frac{1}{2}(2 + 6)\right)$$
$$= \left(3\frac{1}{2}, 4\right)$$

To find the coordinates of D, first plot the information on a graph:



The point D lies on the diagonal BD and is the same distance from M as B is from M. These conditions give two equations which can be solved to find the coordinates of D.

$$distance\ from\ B\ to\ M = \sqrt{(3 - 3.5)^2 + (5 - 4)^2}$$
$$= \sqrt{0.25 + 1}$$
$$= \sqrt{1.25}$$

Equation of the diagonal BD:

$$gradient = \frac{4 - 5}{3.5 - 3} = -\frac{1}{0.5} = -2$$

The equation of the diagonal BD is $y = -2x + c$. At $(3,5)$:

$$5 = -2(3) + c$$

$$c = 11$$

Equation of BD:

$$y = -2x + 11$$

So the point $D\ (x, y)$ satisfies the following equations:

$$\sqrt{(x-3.5)^2 + (y-4)^3} = \sqrt{1.25} \qquad (1)$$

$$y = -2x + 11 \qquad (2)$$

Starting with eqn (1):

$$(x-3.5)^2 + (y-4)^2 = 1.25 \qquad (3)$$

(2) → (3):

$$(x-3.5)^2 + (-2x+7)^2 = 1.25$$

$$x^2 - 7x + 12.25 + 4x^2 - 28x + 49 = 1.25$$

$$5x^2 - 35x + 60 = 0$$

$$x^2 - 7x + 12 = 0$$

$$(x-3)(x-4) = 0$$

$$x = 3\ or\ x = 4$$

The x coordinate of D is 4 since the x coordinate of B is 3. So

$$y = -2(4) + 11 = 3$$

The coordinates of D are $(4,3)$.

12a) The line AP is perpendicular to the line $y = 3x$. So the gradient of AP is $-\frac{1}{3}$. The equation of AP is $y = -\frac{1}{3}x + c$. At $(0,3)$:

$$3 = -\frac{1}{3}(0) + c$$

$$c = 3$$

So the equation of AP is $y = -\frac{1}{3}x + 3$.

b) The lines AP and $y = 3x$ intersect at the point P:

$$3x = -\frac{1}{3}x + 3$$

$$9x + x = 9$$

$$x = 0.9$$

Solve for y:

$$y = 3(0.9)$$

$$y = 2.7$$

The coordinates of P are $(0.9,2.7)$.

c) The perpendicular distance of A from the line $y = 3x$ is the distance between the point A and the point P:

$$distance = \sqrt{(0.9-0)^2 + (2.7-3)^2}$$

$$= \sqrt{0.9^2 + 0.3^2}$$

$$= \sqrt{0.9}$$

13) To show that the given points are collinear, we must show that one of the points lies on the line passing through the other two points.

Equation of the line passing through $(-11,-5)$ and $(4,7)$:

$$gradient = \frac{7-(-5)}{4-(-11)} = \frac{12}{15} = \frac{4}{5}$$

Equation of line: $y = \frac{4}{5}x + c$. At $(4,7)$:

$$7 = \frac{4}{5}(4) + c$$

$$c = 7 - \frac{16}{5} = \frac{19}{5}$$

So the equation of the line passing through $(-11,-5)$ and $(4,7)$ is $y = \frac{4}{5}x + \frac{19}{5}$.

To see if this line passes through the point $(-1,3)$ find the value of y when $x = -1$:

$$y = \frac{4}{5}(-1) + \frac{19}{5}$$

$$= \frac{15}{5} = 3$$

So this line does indeed pass through the point $(-1,3)$. So the given points are collinear.

14i) To find the value of k substitute the coordinates of the point A into the equation of the line:

$$4x + ky = 20$$

$$4(8) + k(-4) = 20$$

$$32 - 4k = 20$$

$$4k = 12$$

$$k = 3$$

So the equation of the line is $4x + 3y = 20$.

Now we can find the value of b, by substituting $(b, 2b)$ into the equation of the line:

$$4(b) + 3(2b) = 20$$

$$10b = 20$$

$$b = 2$$

ii)

$$midpoint = \left(\frac{1}{2}(8+2), \frac{1}{2}(-4+4)\right) = (5,0)$$

[Note: The answer at the back of the text book is incorrect.]

15) The point D is the point of intersection of the lines AD and DC. Since AD is perpendicular to AB and the coordinates of both A and B are given, we can find the equation of AD:

$$gradient\ of\ AB = \frac{8-2}{3-6} = \frac{6}{-3} = -2$$

So the gradient of AD is $\frac{1}{2}$. The equation of AD is $y = \frac{1}{2}x + c$. At $(3,8)$:

$$8 = \frac{1}{2}(3) + c$$

$$c = 8 - \frac{3}{2} = \frac{13}{2}$$

So the equation of AD is $y = \frac{1}{2}x + \frac{13}{2}$.

The line DC is parallel to AB, so its gradient is -2. The equation of DC is $y=-2x+c$. At $(10,2)$:

$$2=-2(10)+c$$
$$c=2+20=22$$

So the equation of DC is $y=-2x+22$.

Now the point D is the intersection of these two lines. So we can equate the right hand sides of the two equations and solve for x:

$$\frac{1}{2}x+\frac{13}{2}=-2x+22$$
$$x+4x=44-13$$
$$5x=31$$
$$x=\frac{31}{5}=6\frac{1}{5}$$
$$\therefore\quad y=-2\left(\frac{31}{5}\right)+22$$
$$y=\frac{48}{5}=9\frac{3}{5}$$

So the coordinates of D are $\left(6\frac{1}{5},9\frac{3}{5}\right)$.

[Note: The answer at the back of the text book is incorrect.]

17) The point B is the intersection of the line segments AC and BD. The equation of the line segment BD can be obtained since it is perpendicular to AC and the coordinates of the point D are given.

Rearrange the equation of AC:

$$2y=x+4 \qquad (1)$$
$$y=\frac{1}{2}x+2$$

So the gradient of BD is -2. The equation of BD is $y=-2x+c$. At $(10,-3)$:

$$-3=-2(10)+c$$
$$c=-3+20=17$$

So the equation of BD is

$$y=-2x+17 \qquad (2)$$

Now find the point at which AC intersects BD:

(2) → (1):

$$2(-2x+17)=x+4$$
$$-4x-x=4-34$$
$$-5x=-30$$
$$x=6$$
$$\therefore\quad y=-2(6)+17$$
$$y=5$$

So the coordinates of B are $(6,5)$.

The point C lies on the line $2y=x+4$ and is the same distance from the point B as the point A is from B. Thus, to find the coordinates of C we first need to find the coordinates of A and the distance from A to B.

The point A lies on the y-axis, so its x coordinate is zero. We can find the y coordinate by substituting $x=0$ into eqn (1):

$$2y=4$$

$$y = 2$$

So the coordinates of A are $(0,2)$.

$$distance\ from\ B\ to\ A = \sqrt{(6-0)^2 + (5-2)^2}$$
$$= \sqrt{36+9} = \sqrt{45}$$

So the point $C\ (x, y)$ satisfies the following equations:

$$(x-6)^2 + (y-5)^2 = 45 \qquad (1)$$

$$y = \frac{1}{2}x + 2 \qquad (2)$$

(2) → (1):

$$(x-6)^2 + \left(\frac{1}{2}x - 3\right)^2 = 45$$
$$x^2 - 12x + 36 + \frac{1}{4}x^2 - 3x + 9 = 45$$
$$\frac{5}{4}x^2 - 15x = 0$$
$$x^2 - 12x = 0$$
$$x(x-12) = 0$$

$$x = 0\ or\ x = 12$$

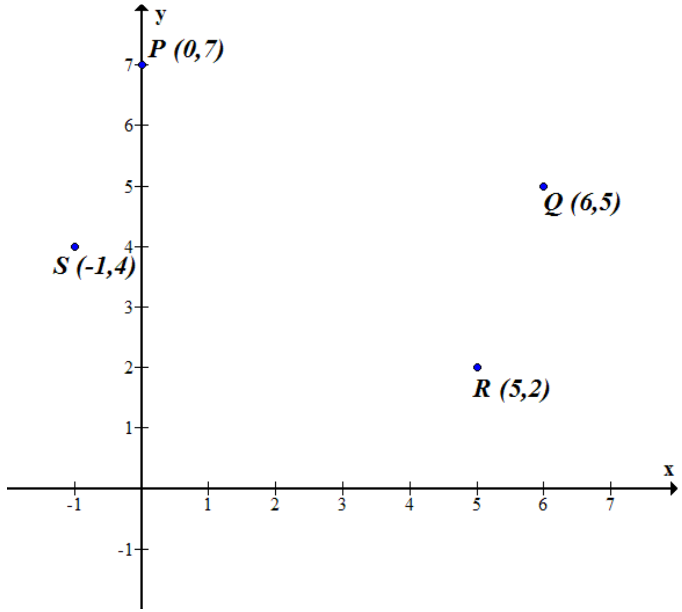
The x coordinate of C is 12, since the x coordinate of A is zero. So we can substitute this value of x into eqn (2) to solve for y:

$$y = \frac{1}{2}(12) + 2$$
$$y = 6 + 2 = 8$$

So the coordinates of the point C are $(12,8)$.

[Note: The answer at the back of the text book is incorrect.]

18) Plot the given points on a coordinate plane:



A rectangle has two sets of parallel lines, each set having the same length and all its corners are right angles. To show that the given points satisfy these conditions we must show that the distance between S and P is the same as the distance between R and Q and that the distance between R and S is the same the distance between P and Q. We must also show that RS is parallel to QP and that SP is parallel to RQ and that SP and RQ are perpendicular to RS and PQ. We start by finding the distances between the points:

$$distance\ from\ S\ to\ P = \sqrt{(-1-0)^2 + (4-7)^2} = \sqrt{10}$$

$$distance\ from\ R\ to\ Q = \sqrt{(5-6)^2 + (2-5)^2} = \sqrt{10}$$

$$distance\ from\ S\ to\ R = \sqrt{(-1-5)^2 + (4-2)^2} = \sqrt{40}$$

$$distance\ from\ P\ to\ Q = \sqrt{(0-6)^2 + (7-5)^2} = \sqrt{40}$$

So we see that

$$distance\ from\ S\ to\ P = distance\ from\ R\ to\ Q$$

$$distance\ from\ S\ to\ R = distance\ from\ P\ to\ Q$$

Next we find the gradients of the line segments joining the points:

$$gradient\ of\ SP = \frac{4-7}{-1-0} = 3$$

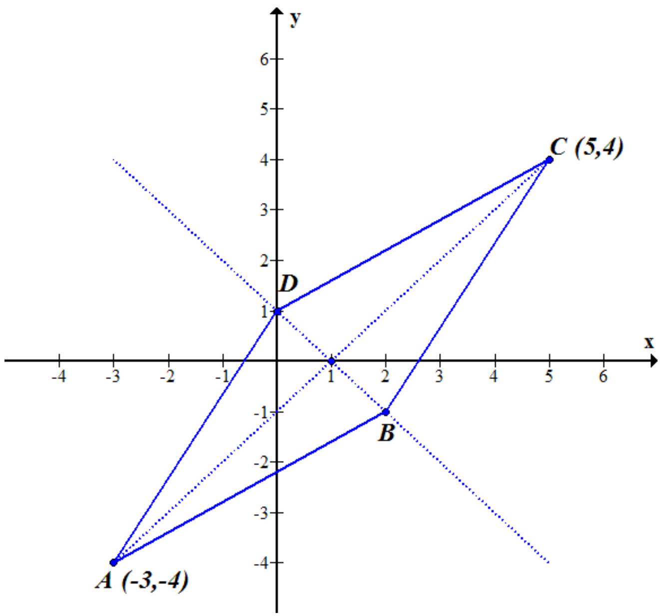
$$gradient\ of\ RP = \frac{5-2}{6-5} = 3$$

$$gradient\ of\ SR = \frac{4-2}{-1-5} = -\frac{1}{3}$$

$$gradient\ of\ PQ = \frac{7-5}{0-6} = -\frac{1}{3}$$

So it is clear that SP and RP are parallel and SR and PQ are parallel. It is also clear that SP and RP are perpendicular to SR and PQ. So the parallel line segments have the same length and all the corners are right angled. Thus these points do form a rectangle.

20a) The diagonals of a rhombus bisect each other at $90°$, but unlike a square they are not the same length. Start by plotting the points A and C and completing a rhombus using arbitrary points for B and D. Note: at this stage we are not certain whether or not D lies on the y-axis.



Since we are given the points A and C we can find the midpoint of AC and then the equation of the diagonal BD:

$$midpoint\ of\ AC = \left(\frac{1}{2}(-3+5), \frac{1}{2}(-4+4)\right) = (1,0)$$

$$gradient\ of\ AC = \frac{-4-4}{-3-5} = 1$$

BD is perpendicular to AC so the gradient of BD is -1. The equation of BD is $y = -x + c$. At $(1,0)$:

$$0 = -1(1) + c$$

$$c = 1$$

So the equation of the diagonal BD is $y = -x + 1$.

b) The point B is the intersection of the lines BD and BC. We are given the gradient of BC and the coordinates of C so we can find the equation of BC. The equation is $y = \frac{5}{3}x + c$. At $(5,4)$:

$$4 = \frac{5}{3}(5) + c$$

$$c = 4 - \frac{25}{3} = -\frac{13}{3}$$

So the equation of BC is $y = \frac{5}{3}x - \frac{13}{3}$.

Next we find the point of intersection of BD and BC:

$$-x + 1 = \frac{5}{3}x - \frac{13}{3}$$

$$5x + 3x = 3 + 13$$

$$8x = 16$$

$$x = 2$$

$$\therefore \quad y = -2 + 1 = -1$$

The coordinates of B are $(2, -1)$.

The point D is the intersection of the line segments BD and AD. Since AD is parallel to BC we can find the equation of AD. It is given by $y = \frac{5}{3}x + c$. At $A\ (-3, -4)$:

$$-4 = \frac{5}{3}(-3) + c$$

$$c = -4 + 5 = 1$$

So the equation of AD is $y = \frac{5}{3}x + 1$.

Now we find the point of intersection between AD and BC:

$$-x + 1 = \frac{5}{3}x + 1$$

$$3x + 5x = 0$$

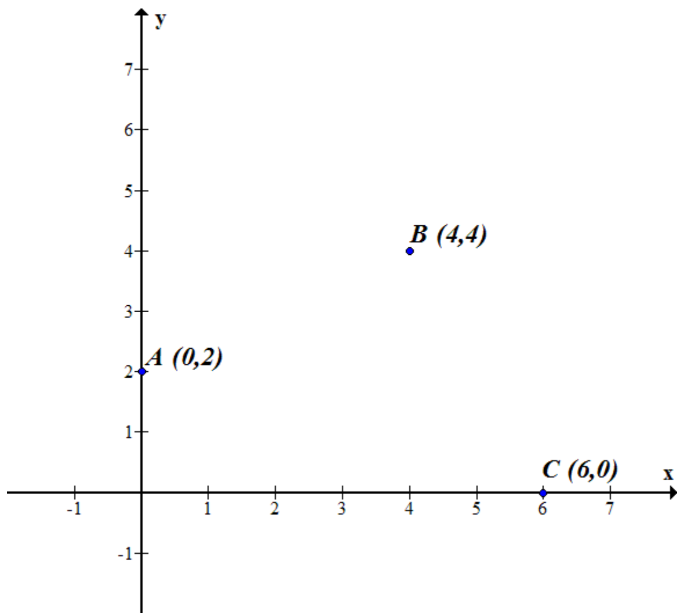
$$x = 0$$

$$\therefore \quad y = -(0) + 1 = 1$$

The coordinates of D are $(0,1)$.

21) Note: a median is the line joining a vertex of a triangle to the midpoint of the opposite side.

First plot and label the points:



Next we find the midpoints of each side:

$$M_1 = midpoint\ of\ AB = \left(\frac{1}{2}(0+4), \frac{1}{2}(2+4)\right) = (2,3)$$

$$M_2 = midpoint\ of\ BC = \left(\frac{1}{2}(4+6), \frac{1}{2}(4+0)\right) = (5,2)$$

$$M_3 = midpoint\ of\ AC = \left(\frac{1}{2}(0+6), \frac{1}{2}(2+0)\right) = (3,1)$$

Now find the equation of each median:

$$gradient\ of\ AM_2 = \frac{2-2}{0-5} = 0$$

The equation of AM_2 is $y = c$. Since the line passes through $(0,2)$, the value of c is 2. So the equation of AM_2 is

$$y = 2 \qquad (1)$$

$$gradient\ of\ BM_3 = \frac{4-1}{4-3} = 3$$

The equation of BM_3 is $y = 3x + c$. At $(4,4)$:

$$4 = 3(4) + c$$

$$c = 4 - 12 = -8$$

So the equation of BM_3 is

$$y = 3x - 8 \qquad (2)$$

$$gradient\ of\ CM_1 = \frac{0-3}{6-2} = -\frac{3}{4}$$

The equation of CM_1 is $y = -\frac{3}{4}x + c$. At $(6,0)$:

$$0 = -\frac{3}{4}(6) + c$$

$$c = \frac{9}{2}$$

So the equation of CM_1 is

$$y = -\frac{3}{4}x + \frac{9}{2} \qquad (3)$$

To show that all the medians are concurrent, we find the point of intersection of AM_2 and BM_3 and show that CM_1 passes through this point:

(1) → (2):

$$2 = 3x - 8$$

$$3x = 10$$

$$x = \frac{10}{3}$$

So AM_2 and BM_3 intersect at $\left(\frac{10}{3}, 2\right)$. To show that CM_1 passes through $\left(\frac{10}{3}, 2\right)$, we find the y value of CM_1 when $x = \frac{10}{3}$.

$$y = -\frac{3}{4}\left(\frac{10}{3}\right) + \frac{9}{2} = \frac{4}{2} = 2$$

So the medians are concurrent and all pass through the point $(\frac{10}{3}, 2)$.