PURE MATHEMATICS 1

WORKED SOLUTIONS FOR CHAPTERS 1 TO 3 OF

Pure Mathematics 1: Coursebook

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This is a free sample comprising chapters 1 to 3 of the complete worked solutions. Solutions for all the chapters may be purchased from https://imago-education.com/products-and-services/as-maths-code-9709-p1.html

NOTATION AND TERMINOLOGY

strict inequality	one which doesn't include 'equal to' $<$ and $>$ are strict inequalities
weak inequality	one which includes 'equal to' $\leq$ and $\geq$ are weak inequalities
$\therefore$	*therefore*
$\because$	*because*, or *since*
eqn	*equation*
$\rightarrow$	*substitute into* E.g (1) $\rightarrow$ (2) means 'substitute equation (1) into equation (2)'.
$\rightarrow$	*tends to* or *approaches*
$\Rightarrow$	*implies* E.g. A $\Rightarrow$ B means 'statement A implies statement B'. In other words, if statement A is true, statement B must be true (but not necessarily the other way around).
$\Leftarrow$	*is implied by* E.g. A $\Leftarrow$ B means 'statement A is implied by statement B'. In other words, if statement B is true, statement A must be true (but not necessarily the other way around).
$\Leftrightarrow$	*implies and is implied by* E.g. A $\Leftrightarrow$ B means 'statement A implies and is implied by statement B'. In other words, if statement A is true, statement B must be true AND if statement B is true, statement A must be true.

TBP	*to be proved* (used at beginning of proof)
QED	*which was to be shown* (used at end of proof) (from Latin *quod erat demonstrandum*)
Δ	*change in*
Δ	The discriminant ($= b^2 - 4ac$) of a quadratic expression in the form $ax^2 + bx + c$.
	Note: the context will indicate which use of Δ is applicable in any particular situation.
δ	*a small change in*
$\perp$	*perpendicular to*
$\equiv$	*identical to*
$\forall$	*for all*
$\exists$	*there exists*

CHAPTER 1

EXERCISE 1A

Question 1

b)

$$length = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{\left(1-(-3)\right)^2 + (-1-2)^2}$$
$$= \sqrt{4^2 + 3^2}$$
$$= \sqrt{16+9}$$
$$= \sqrt{25}$$
$$= 5$$

d)

$$length = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{\left(-7-(-3)\right)^2 + \left(3-(-3)\right)^2}$$
$$= \sqrt{(-4)^2 + 6^2}$$
$$= \sqrt{16+36}$$
$$= \sqrt{52}$$
$$= \sqrt{4\times 13} = \sqrt{4}\sqrt{13} = 2\sqrt{13}$$

f)

$$length = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{\left((a-1)-(a+1)\right)^2 + \left((2a-1)-(2a+3)\right)^2}$$
$$= \sqrt{(-2)^2 + (-4)^2}$$
$$= \sqrt{4+16} = \sqrt{20}$$

h)

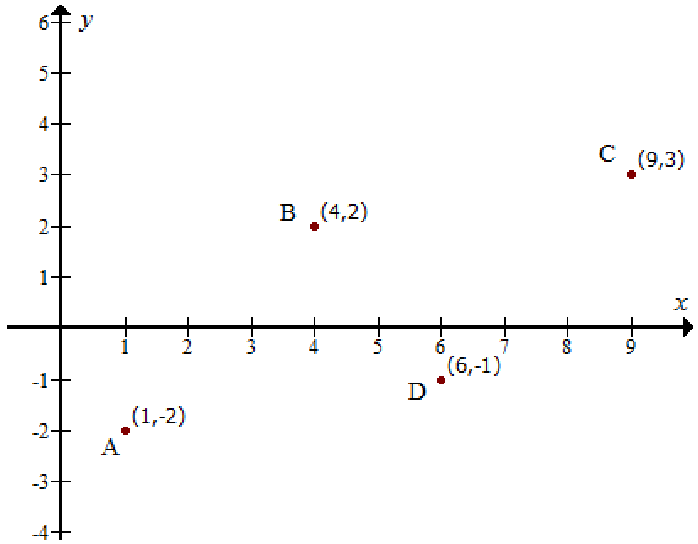
$$length = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(3a-12a)^2 + (5b-5b)^2}$$
$$= \sqrt{(-9a)^2 + 0^2}$$
$$= \sqrt{81a^2}$$
$$= 9a$$

j)

$$length = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{\left((p-3q)-(p+4q)\right)^2 + (p-(p-q))^2}$$
$$= \sqrt{(-7q)^2 + q^2}$$
$$= \sqrt{50q^2}$$
$$= q\sqrt{50}$$

Question 2

Plot the points on the co-ordinate plane:



Now show that a pair of opposite sides is parallel and equal in length.

Take AB, formed by (1,-2) and (4,2):

$$gradient = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2-(-2)}{4-1}$$
$$= \frac{4}{3}$$

$$length = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(4-1)^2 + \big(2-(-2)\big)^2}$$
$$= \sqrt{3^2 + 4^2} = 5$$

Now take DC, formed by (6,-1) and (9,3):

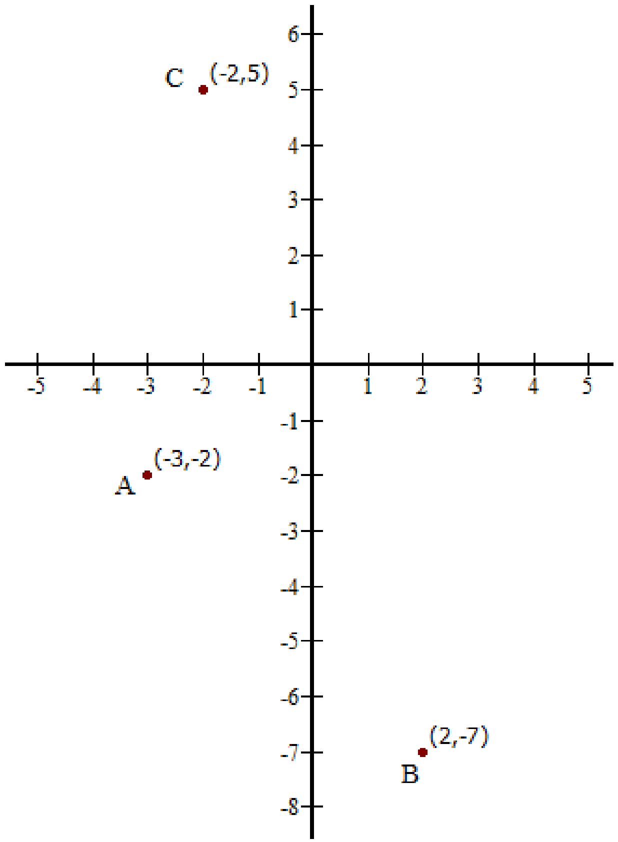
$$gradient = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{3-(-1)}{9-6}$$
$$= \frac{4}{3}$$

$$length = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(9-6)^2 + \big(3-(-1)\big)^2}$$
$$= \sqrt{3^2 + 4^2} = 5$$

Therefore AB and DC are parallel and equal in length, so the four points form a parallelogram.

Question 3

Plot the three points:



From the diagram it appears most likely that sides AC and AB will be equal. Therefore calculate their lengths:

$$length\ AB = \sqrt{\left(2-(-3)\right)^2 + \left(-7-(-2)\right)^2}$$

$$= \sqrt{5^2+5^2} = \sqrt{50}$$

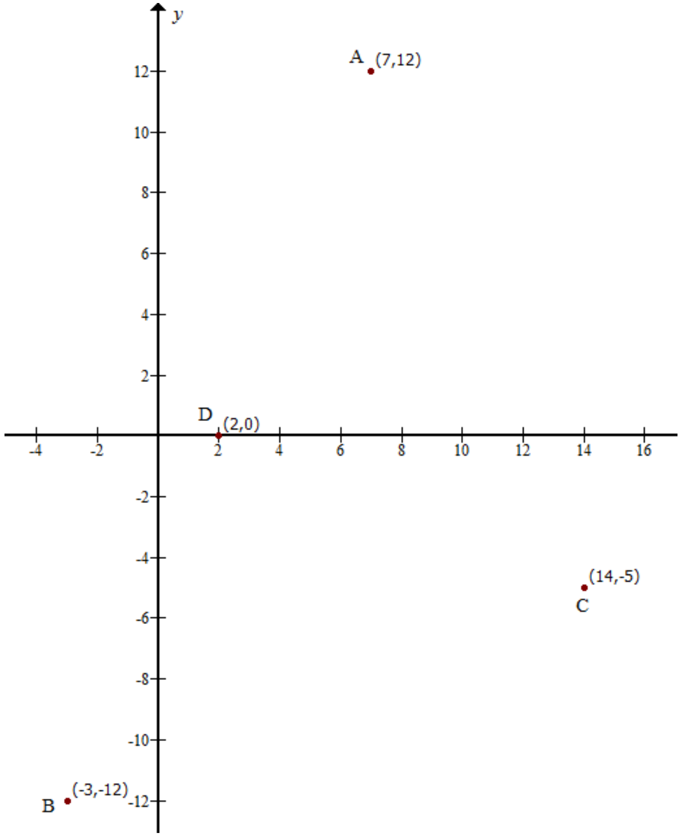
$$length\ AC = \sqrt{\left(-2-(-3)\right)^2 + \left(5-(-2)\right)^2}$$

$$= \sqrt{1^2+7^2} = \sqrt{50}$$

Therefore AB = AC, which means that triangle ABC is isosceles.

Question 4

Plot the points and the centre of the circle:



To show that A, B & C lie on a circle with centre D, we must prove that AD = BD = CD:

$$AD = \sqrt{(7-2)^2 + (12-0)^2}$$
$$= \sqrt{5^2 + 12^2} = 13$$

$$BD = \sqrt{\big(2-(-3)\big)^2 + \big(0-(-12)\big)^2}$$
$$= \sqrt{5^2 + 12^2} = 13$$

$$CD = \sqrt{(14-2)^2 + (-5-0)^2}$$
$$= \sqrt{12^2 + 5^2} = 13$$

Therefore AD = BD = CD, which means that A, B, C all lie on a circle with centre D.

(Notice that, when calculating the length of a line segment, one can use either point as the first one. Thus, in calculating CD above, D was used as the first point and C as the second, giving $(14-2)$ and $(-5-0)$ rather than $(2-14)$ and $(0-(-5))$. The result is the same either way because $12^2 = (-12)^2$.)

Question 5

a) Midpoint is $(x, y) = \left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right)$:

$$x = \frac{1}{2}(2+6) = 4$$
$$y = \frac{1}{2}(11+15) = 13$$

Therefore the midpoint is $(4{,}13)$.

c)

$$x = \frac{1}{2}(-2+1) = -\frac{1}{2}$$
$$y = \frac{1}{2}\big(-3+(-6)\big) = -\frac{9}{2}$$

Therefore midpoint is $\left(-\frac{1}{2}, -\frac{9}{2}\right)$.

e)

$$x = \frac{1}{2}(p+2+3p+4) = \frac{1}{2}(4p+6) = 2p+3$$
$$y = \frac{1}{2}(3p-1+p-5) = \frac{1}{2}(4p-6) = 2p-3$$

Midpoint is $(2p+3, 2p-3)$.

g)

$$x = \frac{1}{2}(p+2q+5p-2q) = \frac{1}{2}(6p) = 3p$$
$$y = \frac{1}{2}\big(2p+13q+(-2p-7q)\big) = \frac{1}{2}(6q) = 3q$$

Midpoint is $(3p, 3q)$.

Question 6

The centre of the circle (let's call it C) is at the midpoint of AB.

$$x_C = \frac{1}{2}(-2+6) = 2$$
$$y_C = \frac{1}{2}(1+5) = 3$$
$$C = (2, 3)$$

Question 7

We use the midpoint formula to set up two equations, which we solve for x_B and y_B:

$$\begin{aligned} x_M &= \frac{1}{2}(x_A + x_B) \\ 5 &= \frac{1}{2}(3 + x_B) \\ 10 &= 3 + x_B \\ x_B &= 7 \end{aligned}$$

$$\begin{aligned} y_M &= \frac{1}{2}(y_A + y_B) \\ 7 &= \frac{1}{2}(4 + y_B) \\ 14 &= 4 + y_B \\ y_B &= 10 \end{aligned}$$

Therefore $B = (7,10)$.

Question 10

a)

$$\begin{aligned} gradient &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{12-8}{5-3} = \frac{4}{2} = 2 \end{aligned}$$

c)

$$gradient = \frac{-1-(-3)}{0-(-4)} = \frac{2}{4} = \frac{1}{2}$$

e)

$$\begin{aligned} gradient &= \frac{(-p-5)-(p-3)}{(2p+4)-(p+3)} = \frac{-2p-2}{p+1} \\ &= \frac{-2(p+1)}{p+1} = -2 \end{aligned}$$

g)

$$\begin{aligned} gradient &= \frac{(q-p+3)-(q+p-3)}{(p-q+1)-(p+q-1)} = \frac{-2p+6}{-2q+2} \\ &= \frac{-2(p-3)}{-2(q-1)} = \frac{p-3}{q-1} \end{aligned}$$

Question 11

$$gradient_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{6-4}{7-3} = \frac{2}{4} = \frac{1}{2}$$

$$gradient_{BC} = \frac{y_C - y_B}{x_C - x_B} = \frac{1-6}{-3-7} = \frac{-5}{-10} = \frac{1}{2}$$

The gradients of AB and BC are equal, which means that the lines AB and BC are parallel. Since they have point B in common, A, B and C must be on the same straight line (i.e. they are collinear).

Question 12

$$gradient_{AP} = \frac{y_P - y_A}{x_P - x_A} = \frac{y-0}{x-3} = \frac{y}{x-3}$$

$$gradient_{PB} = \frac{y_B - y_P}{x_B - x_P} = \frac{6-y}{5-x}$$

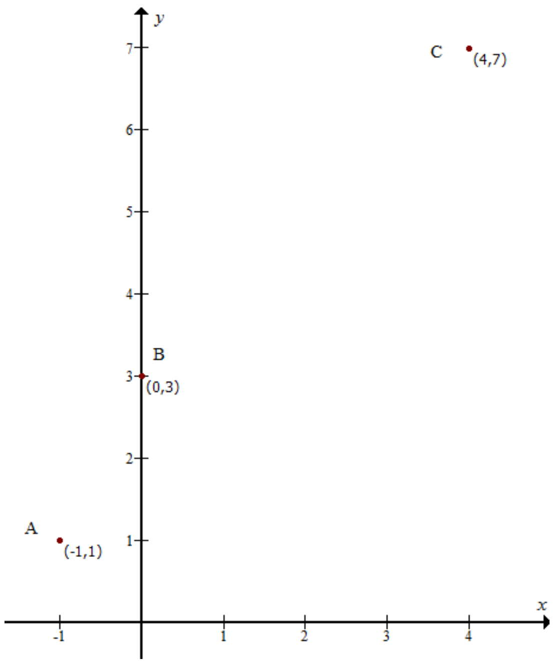
Since A, P & B are all on the same straight line, $gradient_{AP} = gradient_{PB}$. Therefore

$$\begin{aligned} \frac{y}{x-3} &= \frac{6-y}{5-x} \\ y(5-x) &= (6-y)(x-3) \\ 5y - xy &= 6x - 18 - xy + 3y \\ 2y &= 6x - 18 \end{aligned}$$

$$y = 3x - 9$$

Question 13

Draw the points:



(Note: it looks like the points are almost on the same straight line, rather than forming a triangle. Geometrically, this can be a little confusing, but the algebraic calculations are exactly the same.)

The median AM will join A and M, with M being the midpoint of BC (since BC is the side opposite to A). We must first find the coordinates of M using the midpoint formula:

$$x_M = \frac{1}{2}(x_B + x_C) = \frac{1}{2}(0 + 4) = 2$$

$$y_M = \frac{1}{2}(y_B + y_C) = \frac{1}{2}(3 + 7) = 5$$

$$\therefore M = (2,5)$$

Now we can find the length of AM using the distance formula:

$$length_{AM} = \sqrt{(x_M - x_A)^2 + (y_M - y_A)^2}$$

$$= \sqrt{\big(2 - (-1)\big)^2 + (5 - 1)^2}$$

$$= \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$$

Question 16

We first find the coordinates of P, Q, R & S using the midpoint formula:

$$x_P = \frac{1}{2}(1 + 7) = 4 \qquad x_Q = \frac{1}{2}(7 + 9) = 8$$

$$y_P = \frac{1}{2}(1 + 3) = 2 \qquad y_Q = \frac{1}{2}\big(3 + (-7)\big) = -2$$

$$\therefore P = (4,2) \qquad \therefore Q = (8,-2)$$

$$x_R = \frac{1}{2}\big(9 + (-3)\big) = 3 \qquad x_S = \frac{1}{2}(-3 + 1) = -1$$

$$y_R = \frac{1}{2}\big(-7 + (-3)\big) = -5 \qquad y_S = \frac{1}{2}(-3 + 1) = -1$$

$$\therefore R = (3,-5) \qquad \therefore S = (-1,-1)$$

Now we calculate the gradients of the four sides using the gradient formula with the coordinates of the points just calculated:

$$gradient_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{-2-2}{8-4} = \frac{-4}{4} = -1$$

$$gradient_{QR} = \frac{-5-(-2)}{3-8} = \frac{-3}{-5} = \frac{3}{5}$$

$$gradient_{RS} = \frac{-1-(-5)}{-1-3} = \frac{4}{-4} = -1$$

$$gradient_{SP} = \frac{2-(-1)}{4-(-1)} = \frac{3}{5}$$

Thus, the opposite sides PQ and RS are parallel, and the opposite sides QR and SP are parallel; therefore the quadrilateral PQRS is a parallelogram.

Question 18

a) We use the distance formula:

$$ON = \sqrt{(6-0)^2 + (4-0)^2} = \sqrt{6^2+4^2} = \sqrt{60}$$

$$LM = \sqrt{\left(4-(-2)\right)^2 + (7-3)^2} = \sqrt{6^2+4^2} = \sqrt{60}$$

Thus $ON = LM$ (qed).

b) We calculate gradients using the gradient formula:

$$gradient_{ON} = \frac{4-0}{6-0} = \frac{2}{3}$$

$$gradient_{LM} = \frac{7-3}{4-(-2)} = \frac{4}{6} = \frac{2}{3}$$

Therefore ON is parallel to LM (since their gradients are equal).

c) We use the distance formula:

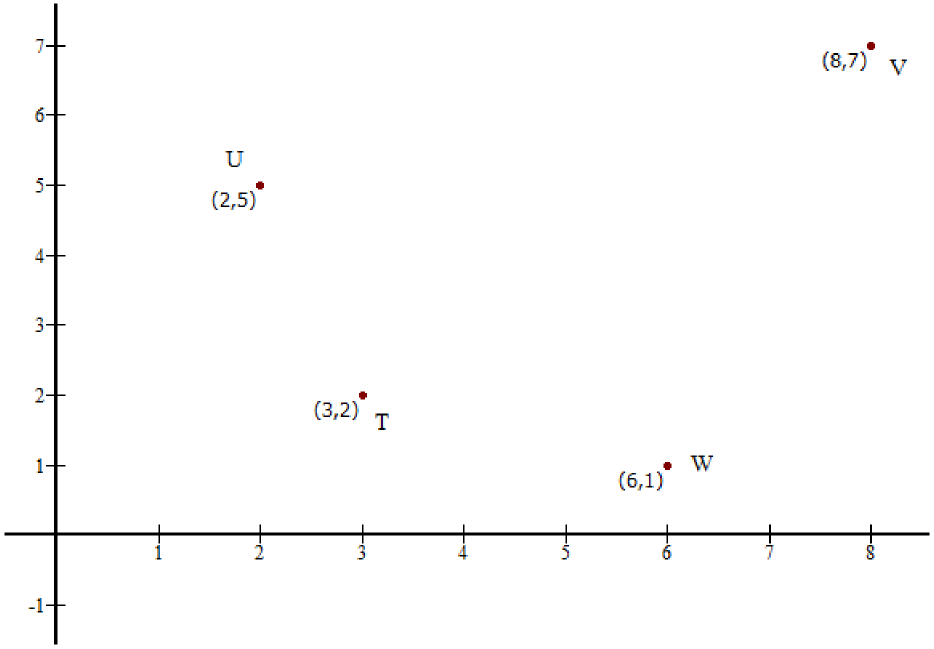
$$OM = \sqrt{(4-0)^2 + (7-0)^2} = \sqrt{4^2+7^2} = \sqrt{16+49} = \sqrt{65}$$

$$LN = \sqrt{\left(6-(-2)\right)^2 + (4-3)^2} = \sqrt{8^2+1^2} = \sqrt{65}$$

d) OLMN is a quadrilateral with two opposite sides (ON and LM) equal and parallel. This means that OLMN is a parallelogram. In addition, the diagonals OM and LN are equal. Therefore OLMN is a **rectangle**.

Question 20

We start by plotting the points:



We calculate the coordinates of M and N using the midpoint formula:

$$x_M = \frac{1}{2}(x_U + x_V) = \frac{1}{2}(2+8) = 5$$

$$y_M = \frac{1}{2}(y_U + y_V) = \frac{1}{2}(5+7) = 6$$

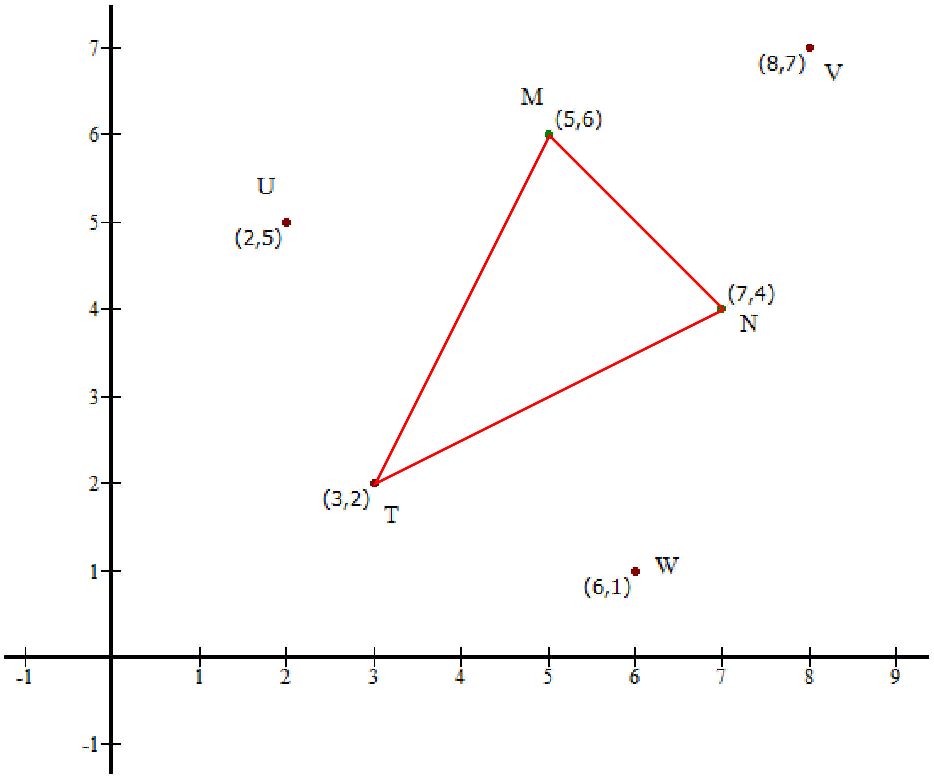
$$\therefore M = (5,6)$$

$$x_N = \frac{1}{2}(x_W + x_V) = \frac{1}{2}(6+8) = 7$$

$$y_N = \frac{1}{2}(y_W + y_V) = \frac{1}{2}(1+7) = 4$$

$$\therefore N = (7,4)$$

To help visualize the points and the triangle TMN, it helps to add the points M and N, and to draw in the triangle:



From the diagram we can see that TM and TN are likely to be the equal sides, therefore we calculate the lengths of these sides using the distance formula:

$$TM = \sqrt{(5-3)^2 + (6-2)^2} = \sqrt{2^2 + 4^2} = \sqrt{20}$$

$$TN = \sqrt{(7-3)^2 + (4-2)^2} = \sqrt{4^2 + 2^2} = \sqrt{20}$$

Therefore TM=TN and triangle TMN is isosceles.

Question 22

a) We start off using the midpoint formula:

$$x_M = \frac{1}{2}(2+10) = 6$$

$$y_M = \frac{1}{2}(1+1) = 1$$

$$\therefore M = (6,1)$$

b) First, we use the distance formula:

$$BG = \sqrt{(6-6)^2 + (4-10)^2} = \sqrt{36} = 6$$

$$GM = \sqrt{(6-6)^2 + (1-4)^2} = \sqrt{9} = 3$$

$$\therefore BG = 2GM$$

To prove that BGM is a straight line, we notice that B, G & M all have the same x-coordinate. Therefore they all lie on the line $x = 6$. Note that we cannot prove this using gradients because a vertical line has an infinite gradient. (Try calculating the gradient of BG, and you will see that you get a division-by-zero error.)

c) We use the midpoint formula:

$$x_N = \frac{1}{2}(6+10) = 8$$

$$y_N = \frac{1}{2}(10+1) = \frac{11}{2}$$

$$\therefore N = (8, \frac{11}{2})$$

d)

$$gradient_{AG} = \frac{4-1}{6-2} = \frac{3}{4}$$

$$gradient_{GN} = \frac{\frac{11}{2} - 4}{8-6} = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4}$$

Therefore A, G, N are all on the same straight line.

Now we use the distance formula:

$$AG = \sqrt{(6-2)^2 + (4-1)^2} = \sqrt{16+9} = 5$$

$$GN = \sqrt{(8-6)^2 + \left(\frac{11}{2} - 4\right)^2} = \sqrt{4 + \frac{9}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

Therefore $AG = 2GN$.

EXERCISE 1B

Question 1

To check whether a point lies on a line, we must check whether the left and right hand sides of the equation are equal when the x and y values of the point are substituted into the equation of the line.

a) $(1,2)$ on $y = 5x - 3$:

$$LHS = 2$$

$$RHS = 5(1) - 3 = 2$$

Therefore the point $(1,2)$ does lie on the line $y = 5x - 3$.

b) $(3, -2)$ on $y = 3x - 7$:

$$LHS = -2$$

$$RHS = 3(3) - 7 = 9 - 7 = 2$$

Therefore the point $(3, -2)$ does not lie on the line $y = 3x - 7$.

d) $(2,2)$ on $3x^2 + y^2 = 40$:

$$LHS = 3(2)^2 + (2)^2 = 3(4) + 4 = 16$$
$$RHS = 40$$

Therefore the point $(2,2)$ does not lie on the line $3x^2 + y^2 = 40$.

f) $(5p, \frac{5}{p})$ on $y = \frac{5}{x}$:

$LHS = \frac{5}{p}$ $\qquad$ $RHS = \frac{5}{5p} = \frac{1}{p}$

Therefore the point $\left(5p, \frac{5}{p}\right)$ does not lie on the line $y = \frac{5}{x}$.

h) $(t^2, 2t)$ on $y^2 = 4x$:

$$LHS = (2t)^2 = 4t^2$$

$$RHS = 4(t^2) = 4t^2$$

Therefore the point $(t^2, 2t)$ does lie on the line $y^2 = 4x$.

Question 2

Use either $y = mx + c$, where m is the gradient of the line and c is the y intercept, or $y - y_1 = m(x - x_1)$. The second is usually easier.

b)

$$m = -3$$
$$y = -3x + c$$
$$y - y_1 = m(x - x_1)$$
$$y - (-2) = -3(x - 1)$$
$$y + 2 = -3x + 3$$
$$y = -3x + 1$$

Therefore the equation is $y = -3x + 1$.

d)

$$y - y_1 = m(x - x_1)$$
$$y - 1 = -\frac{3}{8}(x - (-2))$$
$$y - 1 = -\frac{3}{8}x - \frac{3}{4}$$
$$8y - 8 = -3x - 6$$
$$3x + 8y = 2$$

f) If the gradient is zero (i.e. the line is horizontal), the equation of the line has the form $y = c$. We can take the y-value of any point on the line to get the value of c. Therefore the equation of the line is

$$y = 8$$

h) Use $y = mx + c$:

$$y = \frac{1}{2}x + c$$

Then substitute $(-3,0)$ into eqn.

$$0 = \frac{1}{2}(-3) + c$$
$$0 = c - \frac{3}{2}$$
$$c = \frac{3}{2}$$

Therefore the equation of the line is

$$y = \frac{1}{2}x + \frac{3}{2}$$

or

$$2y = x + 3$$

j)

$$y - y_1 = m(x - x_1)$$
$$y - 4 = -\frac{1}{2}(x - 3)$$
$$2y - 8 = -x + 3$$
$$x + 2y = 11$$

l)

$$y - (-5) = 3\big(x - (-2)\big)$$
$$y + 5 = 3x + 6$$
$$y = 3x + 1$$

n)

$$y = -x + c$$
$$2 = -(0) + c$$
$$c = 2$$

Equation is:

$$y = -x + 2$$

p)

$$y - 0 = -\frac{3}{5}(x - 3)$$
$$5y = -3x + 9$$
$$3x + 5y = 9$$

r)

$$y = mx + c$$
$$4 = m(0) + c$$
$$c = 4$$

Equation is:

$$y = mx + 4$$

t)

$$y = mx + b$$

(Use b instead of c here to avoid confusion with the value of x, which is given as c.)

$$0 = mc + b$$

$$b = -mc$$

Equation is:

$$y = mx - mc$$

Question 3

b)

$$gradient = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7-5}{-2-4} = \frac{-12}{-6} = 2$$

Therefore the equation of the line is $y - y_1 = m(x - x_1)$. Substitute the x and y values of one of the points into this eqn.:

$$y - 5 = 2(x - 4)$$

$$y = 2x - 3$$

f)

$$gradient = \frac{20-(-1)}{-4-3} = \frac{21}{-7} = -3$$

Equation of line: $y - y_1 = m(x - x_1)$. At $(3, -1)$:

$$y + 1 = -3(x - 3)$$

$$\therefore y = -3x + 8$$

j)

$$gradient = \frac{-3-(-1)}{5-(-2)} = \frac{-2}{7} = -\frac{2}{7}$$

Equation of line is $y = -\frac{2}{7}x + c$. At $(-2, -1)$:

$$-1 = -\frac{2}{7}(-2) + c$$

$$-7 = 4 + 7c$$

$$c = -\frac{11}{7}$$

$$\therefore y = -\frac{2}{7}x - \frac{11}{7}$$

$$7y = -2x - 11$$

$$2x + 7y + 11 = 0$$

n)

$$gradient = \frac{-1-4}{-2-(-5)} = \frac{-5}{3} = -\frac{5}{3}$$

Equation of line is $y = -\frac{5}{3}x + c$.

At $(-5, 4)$:

$$4 = -\frac{5}{3}(-5) + c$$

$$12 = 25 + 3c$$

$$3c = -13$$

$$c = -\frac{13}{3}$$

$$y = -\frac{5}{3}x - \frac{13}{3}$$

$$3y = -5x - 13$$

$$5x + 3y + 13 = 0$$

p)

$$gradient = \frac{q-0}{p-0} = \frac{q}{p}$$

Equation of line is $y = \frac{q}{p}x + c$. At $(0,0)$:

$$0 = c$$

$$\therefore y = \frac{q}{p}x$$

$$or \;\; py = qx$$

$$or \;\; qx - py = 0$$

q)

$$gradient = \frac{q-1-q}{p+3-p} = -\frac{1}{3}$$

Equation of line is $y = -\frac{1}{3}x + c$. At (p, q):

$$q = -\frac{1}{3}p + c$$

$$3q = -p + 3c$$

$$3c = 3q + p$$

$$c = \frac{3q+p}{3}$$

$$y = -\frac{1}{3}x + \frac{3q+p}{3}$$

$$3y = -x + 3q + p$$

$$x + 3y - p - 3q = 0$$

r) Notice that both points have the same x-coordinate, p. This means that both points lie on the vertical line, $x = p$. So, the equation of the line joining these points is $x = p$. This question cannot be answered in the same way as above, since the gradient of a vertical line is infinite.

s)

$$gradient = \frac{(q+2)-q}{(p+2)-p}$$

$$= 1$$

Using (p, q):

$$y - q = x - p$$

$$\therefore y = x + q - p$$

$$or \quad x - y + q - p = 0$$

t)

$$gradient = \frac{q-0}{0-p}$$

$$= -\frac{q}{p}$$

Equation of line is $y = -\frac{q}{p}x + c$. At $(0, q)$:

$$q = -\frac{q}{p}(0) + c$$

$$c = q$$

$$\therefore y = -\frac{q}{p}x + q$$

$$or \quad qx + py - pq = 0$$

Question 4

In each case write the given equation in the form $y = mx + c$ and then read off the gradient (it is the coefficient of x).

a)

$$2x + y = 7$$

$$y = -2x + 7$$

From the form of the equation it is now easy to see that the gradient is -2.

c)

$$5x + 2y = -3$$

$$2y = -5x - 3$$

$$y = -\frac{5}{2}x - \frac{3}{2}$$

So, the gradient is $-\frac{5}{2}$.

f) The line $5x = 7$ is a vertical line at $x = \frac{7}{5}$, so the gradient is infinite or undefined.

h)

$$y = 3(x + 4)$$

$$y = 3x + 12$$

Gradient is 3.

k)

$$y = m(x - d)$$

$$y = mx - md$$

Gradient is m.

l)

$$px + qy = pq$$

$$qy = -px + pq$$

$$y = -\frac{p}{q}x + p$$

Gradient is $-\frac{p}{q}$.

Question 5

Since the line must be parallel to the line $y = \frac{1}{2}x - 3$, the gradient is $\frac{1}{2}$.

So the equation of the line is $y = \frac{1}{2}x + c$, and we find the value of c by substituting for x and y at the given point $(-2,1)$:

$$1 = \frac{1}{2}(-2) + c$$

$$c = 1 + 1$$

$$= 2$$

So the equation of the line is $y = \frac{1}{2}x + 2$.

Question 6

First, we find the gradient. The line is parallel to $y + 2x = 7$, which can be written as $y = -2x + 7$. So the gradient is -2.The equation of the line is $y = -2x + c$. At $(4, -3)$:

$$\begin{aligned} -3 &= -2(4) + c \\ c &= -3 + 8 \\ &= 5 \end{aligned}$$

The equation of the line is $y = -2x + 5$.

Question 7

Here the gradient is the same as for the line joining the points $(3, -1)$ and $(-5,2)$:

$$\begin{aligned} gradient &= \frac{2 - (-1)}{(-5) - 3} \\ &= -\frac{3}{8} \end{aligned}$$

The equation of the line is $y = -\frac{3}{8}x + c$. At $(1,2)$:

$$\begin{aligned} 2 &= -\frac{3}{8}(1) + c \\ c &= 2 + \frac{3}{8} \\ &= \frac{19}{8} \end{aligned}$$

The equation of the line is $y = -\frac{3}{8}x + \frac{19}{8}$.

Question 8

Here the gradient is the same as for the line joining the points $(-3,2)$ and $(2, -3)$:

$$\begin{aligned} gradient &= \frac{2 - (-3)}{(-3) - 2} \\ &= -1 \end{aligned}$$

The equation of the line is $y = -x + c$. At $(3,9)$:

$$\begin{aligned} 9 &= -1(3) + c \\ c &= 9 + 3 = 12 \end{aligned}$$

The equation of the line is $y = -x + 12$.

Question 9

A line parallel to the x-axis is a horizontal line with a gradient of zero. Thus, the equation of the line that passes through $(1,7)$ and is parallel to the x-axis is $y = 7$.

Question 10

Since the line must be parallel to $y = mx + c$, the gradient of the line is m. The equation of the line is $y = mx + b$ (using b to avoid confusion). At $(d, 0)$:

$$\begin{aligned} 0 &= md + b \\ b &= -md \end{aligned}$$

The equation of the line is $y = mx - md$.

Question 11

The point of intersection of two lines is the point (x, y) that satisfies both equations.

b) The simplest approach here is to equate the right hand sides of each equation and solve for x:

$$3x + 1 = 4x - 1$$
$$x = 2$$

Now substitute this value of x into either of the given equations:

$$y = 3(2) + 1$$
$$y = 7$$

Check this answer by substituting these values for x and y back into the second equation (the one that was not used to solve for y) and check that both sides are equal:

$$RHS = 4(2) - 1 = 7$$
$$LHS = 7$$
$$\therefore LHS = RHS$$

So the point of intersection is $(2,7)$.

d) Use the same approach as above.

$$3x + 8 = -2x - 7$$
$$5x = -15$$
$$x = -3$$

Substitute for x in the first equation:

$$y = 3(-3) + 8$$
$$y = -1$$

Check:

$$RHS = -2(-3) - 7 = -1$$
$$LHS = -1$$
$$\therefore LHS = RHS$$

So the point of intersection is $(-3, -1)$.

f) Here it is simpler to multiply the first equation by 5 and the second equation by -2 and then to add the equations to eliminate x:

$$10x + 35y = 235$$
$$-10x - 8y = -100$$

Adding these equations and solving for y:

$$27y = 135$$
$$y = 5$$

Substituting for y in the first equation and solving for x:

$$2x + 7(5) = 47$$
$$2x = 12$$
$$x = 6$$

Check by substituting these values for x and y into the second equation:

$$LHS = 5(6) + 4(5) = 30 + 20 = 50 = RHS$$

So the point of intersection is $(6,5)$.

g) Multiply the first equation by -3 and add the two equations to eliminate x:

$$-6x - 9y = -21 \qquad (1)$$

$$6x + 9y = 11 \qquad (2)$$

(1) + (2):

$$0 = -10$$

This is obviously not true. These equations cannot be solved simultaneously, so we conclude that these two lines do not intersect. This can be understood geometrically by writing the equations in the form $y = mx + c$:

$$y = -\frac{2}{3}x + \frac{7}{3}$$

$$y = -\frac{2}{3}x + \frac{11}{9}$$

These lines have the same gradient so they are parallel lines and never intersect. (The y-intercepts are different so they are not collinear. In the case of collinear lines, there are infinitely many points which will simultaneously satisfy both the equations.)

h) Solve the first equation for y:

$$y = -3x + 5$$

Then substitute this expression for y into the second equation and solve for x:

$$x + 3(-3x + 5) = -1$$

$$x - 9x + 15 = -1$$

$$-8x = -16$$

$$x = 2$$

Then solve for y by substituting this value of x into the first expression for y:

$$y = -3(2) + 5$$

$$y = -1$$

Check by substituting these values for x and y into the second equation:

$$LHS = 2 + 3(-1) = -1 = RHS$$

The lines intersect at $(2, -1)$.

i) The first equation is already written as an expression for y, so we just substitute this expression into the second equation:

$$4x - 2(2x + 3) = -6$$

$$4x - 4x - 6 = -6$$

$$-6 = -6$$

This is always true. Since the x's cancel out in the second line, any value of x will simultaneously satisfy these equations. Thus, any value of y will also simultaneously satisfy both these equations. There are thus infinitely many solutions.

This can be seen geometrically as well. Rewriting the equations in the form $y = mx + c$ gives:

$$y = 2x + 3$$

$$y = 2x + 3$$

So the given equations actually represent the same line.

j) The second equation here is already written as an expression for y, so we just substitute that expression into the first equation:

$$ax + b(2ax) = c$$
$$(a + 2ab)x = c$$
$$x = \frac{c}{a + 2ab}$$

To find y we substitute this expression back into the second equation:

$$y = 2a(\frac{c}{a + 2ab})$$
$$y = \frac{2ac}{a + 2ab}$$
$$y = \frac{2c}{1 + 2b}$$

To check we substitute these expressions for x and y into the first equation:

$$LHS = a\left(\frac{c}{a + 2ab}\right) + b\left(\frac{2c}{1 + 2b}\right)$$
$$= \frac{ac + 2abc}{a + 2ab} = \left(\frac{a + 2ab}{a + 2ab}\right)c$$
$$= c = RHS$$

The lines intersect at $(\frac{c}{a+2ab}, \frac{2ac}{a+2ab})$.

k) Add the two equations to eliminate x:

$$2y = c + d$$
$$y = \frac{c + d}{2}$$

Substitute this expression for y into the first equation:

$$\frac{c + d}{2} = mx + c$$
$$mx = \frac{c + d}{2} - c$$
$$x = \frac{c + d - 2c}{2m}$$
$$x = \frac{d - c}{2m}$$

Check by substituting for x and y into the second equation:

$$RHS = -m\left(\frac{d - c}{2m}\right) + d$$
$$= \frac{c - d}{2} + \frac{2d}{2}$$
$$= \frac{c + d}{2} = LHS$$

So the lines intersect at $(\frac{d-c}{2m}, \frac{c+d}{2})$.

l) Substitute the expression for y from the second equation into the first equation:

$$ax - bx = 1$$
$$(a - b)x = 1$$
$$x = \frac{1}{a - b}$$

Substitute this back into the second equation and to obtain y:

$$y = \frac{1}{a-b}$$

Check, using the first equation:

$$LHS = a\left(\frac{1}{a-b}\right) - b\left(\frac{1}{a-b}\right) = \frac{a-b}{a-b} = 1 = RHS$$

So the lines intersect at $\left(\frac{1}{a-b}, \frac{1}{a-b}\right)$.

Question 12

Since $P\ (p.q)$ satisfies the equation $y = mx + c$, we have

$$q = mp + c. \qquad (1)$$

The point $Q\ (r, s)$ is any other point on the line $y = mx + c$ so it also satisfies the equation $y = mx + c$ and we can write

$$s = mr + c. \qquad (2)$$

Now the gradient of the line joining the points P and Q is given as usual by:

$$gradient = \frac{s-q}{r-p}.$$

Substituting for q and s from eqns (1) and (2) gives:

$$gradient = \frac{mr + c - (mp + c)}{r-p}$$

$$gradient = \frac{mr - mp}{r-p}$$

$$gradient = \frac{r-p}{r-p}m$$

$$gradient = m$$

Thus, the gradient of the line segment joining the points P and Q is m. (Since P and Q are arbitrary points on the line, this shows that if you have an equation of a line in the form $y = mx + c$, then the coefficient of x is the gradient of that line, a fact that we used in exercise 4.)

Question 13

If $a = b = c = 0$, then any value of x and y will satisfy the equation $ax + by + c = 0$. This equation then represents the entire xy-plane rather than a straight line.

EXERCISE 1C

Question 1

The gradient of a line that is perpendicular to a given line, is the negative of the reciprocal of the gradient of the given line.

c) $-\frac{4}{3}$

d) $\frac{6}{5}$

g) m

h) $-\frac{1}{m}$

k) $\frac{1}{m}$

l) $-\frac{1}{\frac{a}{b-c}} = -\frac{b-c}{a} = \frac{c-b}{a}$

Question 2

b) The gradient of the given line is $-\frac{1}{2}$, so the gradient of the line is 2. So the equation of the line is $y = 2x + c$. At $(-3,1)$:

$$1 = 2(-3) + c$$
$$c = 7$$
$$\therefore y = 2x + 7$$

d) The line $y = 2\frac{1}{2} = \frac{5}{2}$ is a horizontal line. So, any vertical line is perpendicular to this line. So the equation of the line is $x = 7$.

f) Rewrite the given line in the form $y = mx + c$:

$$3x - 5y = 8$$
$$5y = 3x - 8$$
$$y = \frac{3}{5}x - \frac{8}{5}$$

So the gradient of the perpendicular line is $-\frac{5}{3}$. Using $(4,3)$:

$$y - 3 = -\frac{5}{3}(x - 4)$$
$$3y - 9 = -5x + 20$$
$$5x + 3y = 29$$

h) The gradient is $-\frac{1}{2}$. The equation of the line is $y = -\frac{1}{2}x + c$. At $(0,3)$:

$$3 = -\frac{1}{2}(0) + c$$
$$c = 3$$
$$\therefore \; y = -\frac{1}{2}x + 3$$
$$or \;\; 2y + x = 6$$

j) The gradient is $-\frac{1}{m}$. The equation of the line is $y = -\frac{1}{m}x + d$. At (a, b):

$$b = -\frac{1}{m}a + d$$
$$d = b + \frac{a}{m}$$
$$d = \frac{mb + a}{m}$$

l)

$$ax + by = c$$

$$by = -ax + c$$

$$y = -\frac{a}{b}x + \frac{c}{b}$$

So the gradient of the line is $\frac{b}{a}$. The equation of the line is $y = \frac{b}{a}x + d$. At $(-1,-2)$:

$$-2 = \frac{b}{a}(-1) + d$$

$$d = \frac{b}{a} - 2 = \frac{b-2a}{a}$$

$$\therefore \ y = \frac{b}{a}x + \frac{b-2a}{a}$$

$$or \ \ bx - ay = 2a - b$$

Question 3

The gradient of the line is $-\frac{1}{3}$. The equation of the line is $y = 3x + c$. At $(-2,5)$:

$$5 = -\frac{1}{3}(-2) + c$$

$$c = 5 - \frac{2}{3}$$

$$c = \frac{13}{3}$$

So the equation of the line is $y = -\frac{1}{3}x + \frac{13}{3}$.

To find the point of intersection we equate right hand sides of the two equations:

$$3x + 1 = -\frac{1}{3}x + \frac{13}{3}$$

$$9x + 3 = -x + 13$$

$$10x = 10$$

$$x = 1$$

Using this, we can solve for y:

$$y = 3(1) + 1$$

$$y = 4$$

So the two lines intersect at the point $(1,4)$.

Question 4

Rewrite the given equation in the form $y = mx + c$:

$$2x - 3y = 12$$

$$3y = 2x - 12$$

$$y = \frac{2}{3}x - 4 \qquad (1)$$

So the gradient of the line will be $-\frac{3}{2}$. The equation is $y = -\frac{3}{2}x + c$. At $(1,1)$:

$$1 = -\frac{3}{2}(1) + c$$

$$c = 1 + \frac{3}{2} = \frac{5}{2}$$

$$\therefore \quad y = -\frac{3}{2}x + \frac{5}{2} \qquad (2)$$

To find the point of intersection, substitute eqn (1) into eqn (2):

$$\frac{2}{3}x - 4 = -\frac{3}{2}x + \frac{5}{2}$$
$$4x - 24 = -9x + 15$$
$$13x = 39$$
$$x = 3$$

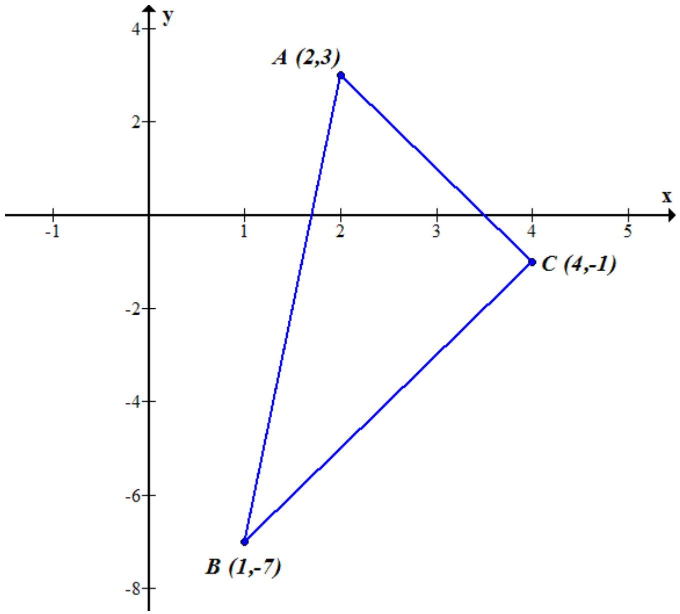
Substitute this result into eqn (1) to solve for y:

$$y = \frac{2}{3}(3) - 4$$
$$y = -2$$

So the lines intersect at $(3, -2)$.

Question 5

First plot the points of the triangle in a coordinate plane:



Now we are looking for the line that is perpendicular to the line segment BC and passes through the point $A\ (2,3)$. So the first step is to find the gradient of the line segment BC:

$$gradient\ of\ BC = \frac{-1-(-7)}{4-1} = \frac{6}{3} = 2$$

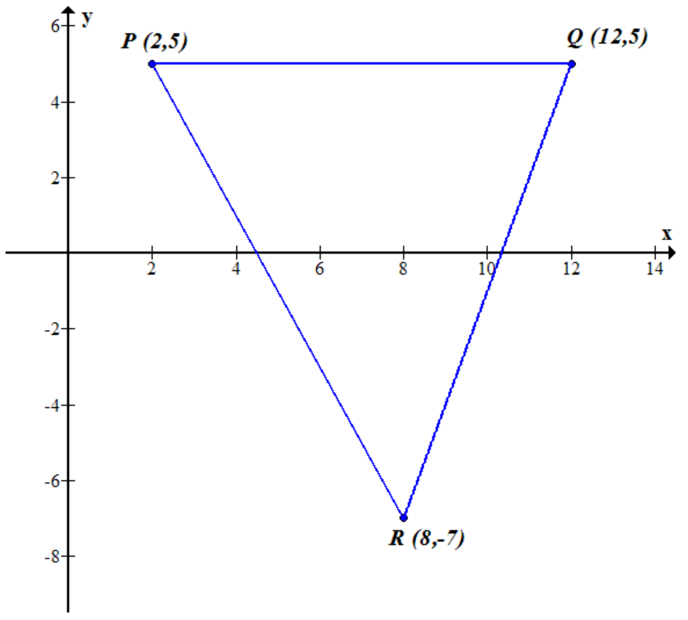
So the gradient of the line we are looking for is $-\frac{1}{2}$. The equation of the line is $y = -\frac{1}{2}x + c$. At $(2,3)$:

$$3 = -\frac{1}{2}(2) + c$$
$$c = 4$$

So the equation of the altitude through the vertex A is $y = -\frac{1}{2}x + 4$.

Question 6

Plot the triangle:



a) The line segment PQ is opposite to the vertex R. Since the line segment PQ is a horizontal line, the altitude through R is a vertical line passing through $(8, -7)$. The equation of this line is $x = 8$.

The altitude through the vertex Q is perpendicular to the line segment PR.

$$gradient\ of\ PR = \frac{5-(-7)}{2-8} = \frac{12}{-6} = -2$$

So the gradient of the altitude through Q is $\frac{1}{2}$. The equation of the line is $y = \frac{1}{2}x + c$. At $Q\ (12,5)$:

$$5 = \frac{1}{2}(12) + c$$

$$c = -1$$

So the equation of the altitude through Q is $y = \frac{1}{2}x - 1$.

b) Find the point of intersection:

$$x = 8 \qquad (1)$$

$$y = \frac{1}{2}x - 1 \qquad (2)$$

(1) → (2):

$$y = \frac{1}{2}(8) - 1 = 3$$

So these two altitudes intersect at the point $(8,3)$.

c) To show that the altitude through P also passes through the point $(8,3)$, we first find the equation of this altitude and then show that the point $(8,3)$ satisfies that equation.

$$gradient\ of\ QR = \frac{5-(-7)}{12-8} = 3$$

So the gradient of the altitude is $-\frac{1}{3}$ and the equation of the line is $y = -\frac{1}{3}x + c$. At $(2,5)$:

$$5 = -\frac{1}{3}(2) + c$$

$$c = 5 + \frac{2}{3} = \frac{17}{3}$$

So the equation of the altitude through P is $y = -\frac{1}{3}x + \frac{17}{3}$.

Now we show that the point $(8,3)$ satisfies this equation by finding the value of y if we substitute $x = 8$ into the equation:

$$y = -\frac{1}{3}(8) + \frac{17}{3}$$

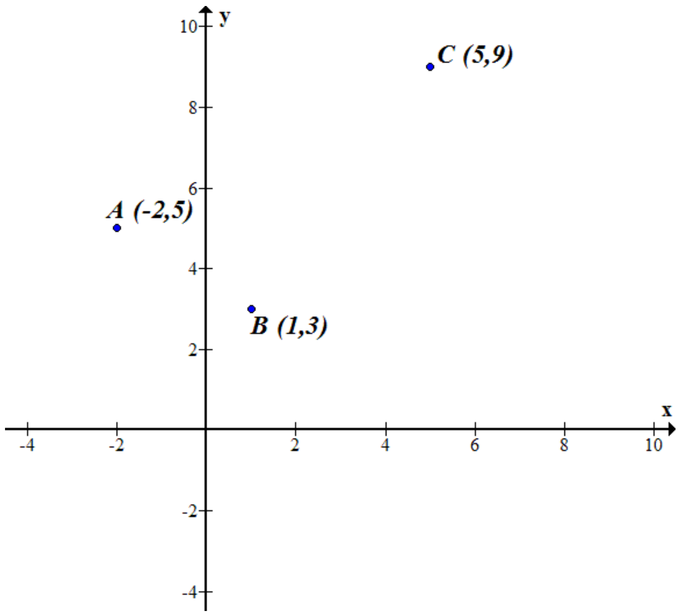
$$y = \frac{9}{3} = 3$$

So the altitude through P also passes through $(8,3)$.

MISCELLANEOUS EXERCISE 1

Question 1

In order to show that a triangle is right-angled, we have to show that two of the line segments joining the vertices are perpendicular. First we plot and label the vertices of the triangle:



From the graph it looks most likely that the line segments AB and BC will be perpendicular. To show this we calculate the gradients of each line segment:

$$gradient\ of\ AB = \frac{3-5}{1-(-2)} = -\frac{2}{3}$$

$$gradient\ of\ BC = \frac{9-3}{5-1} = \frac{6}{4} = \frac{3}{2}$$

We see that the gradient of AB is the negative of the reciprocal of the gradient of BC. So these two line segments are perpendicular.

Question 2

Find the point of intersection:

$$2x + y = 3 \qquad (1)$$

$$3x + 5y - 1 = 0 \qquad (2)$$

Rearrange eqn (1):

$$y = -2x + 3 \qquad (3)$$

(3) → (2):

$$3x + 5(-2x + 3) - 1 = 0$$

$$3x - 10x + 15 - 1 = 0$$

$$7x = 14$$

$$x = 2 \qquad (4)$$

(4) → (3):

$$y = -2(2) + 3$$

$$y = -1 \qquad (5)$$

So the lines intersect at $(2, -1)$.

Question 3

a) To show that the angle ACB is a right angle, we have to show that the line segments AC and CB are perpendicular:

$$gradient\ of\ AC = \frac{8-3}{0-(-1)} = 5$$

$$gradient\ of\ CB = \frac{8-7}{0-5} = -\frac{1}{5}$$

So the gradient of AC is the negative of the reciprocal of the gradient of CB. So these line segments are perpendicular to each other and it follows that the angle between them (angle ACB) is a right angle.

b) First we need to find the equation of the line that is parallel to AC and passes through B. Since it is parallel to AC it has the same gradient as AC which is 5 (from part (a)). So the equation of the line is $y = 5x + c$. At B $(5,7)$:

$$7 = 5(5) + c$$

$$c = 7 - 25 = -18$$

So the equation of the line is $y = 5x - 18$.

To find the point where this line crosses the x-axis, we set y to zero and solve for x:

$$0 = 5x - 18$$

$$5x = 18$$

$$x = 3.6$$

So this line crosses the x-axis at the point $(3.6,0)$.

Question 4

a) The diagonals of a square have the same length and bisect each other at $90°$. So the diagonal BD is perpendicular to AC and passes through the midpoint of AC.

$$midpoint\ of\ AC = \left(\frac{1}{2}(7+1), \frac{1}{2}(2+4)\right)$$
$$= (4,3)$$

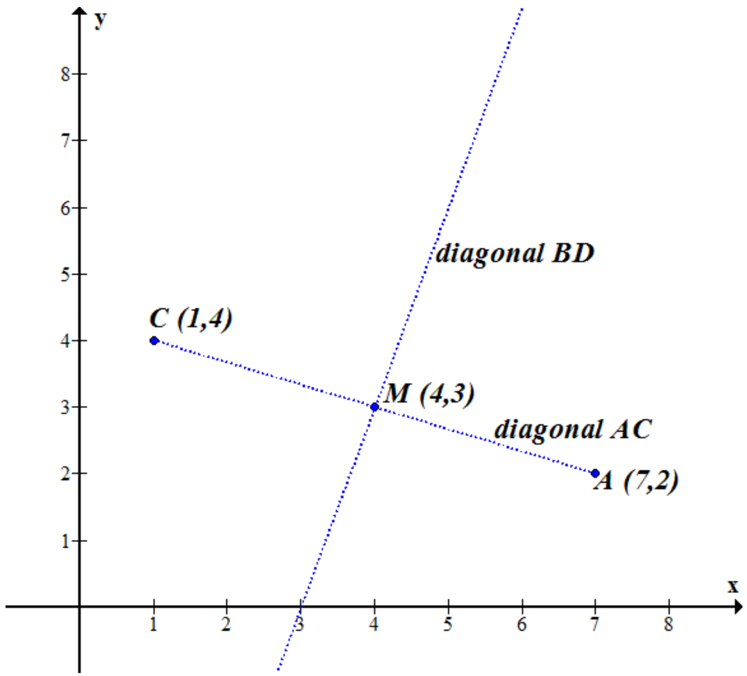
$$gradient\ of\ AC = \frac{4-2}{1-7} = \frac{2}{-6} = -\frac{1}{3}$$

So the gradient of the diagonal BD is 3. The equation of the diagonal is $y = 3x + c$. At $(4,3)$:

$$3 = 3(4) + c$$
$$c = -9$$

So the equation of the diagonal BD is $y = 3x - 9$.

b) Since the diagonals bisect each other and have the same length, the distance from the midpoint of AC to either point B or D is half the distance from point A to point C. Both points B and D also lie on the diagonal BD. First we plot all the information we know:



$$distance\ from\ A\ to\ C = \sqrt{(7-1)^2 + (2-4)^2}$$
$$= \sqrt{40}$$
$$= 2\sqrt{10}$$

So the distance from the midpoint $(4,3)$ to either point B or D is $\sqrt{10}$. Suppose we have a point (x, y) that lies on the diagonal BD and is a distance $\sqrt{10}$ from the midpoint. Then it must satisfy the following equations:

$$\sqrt{(x-4)^2 + (y-3)^2} = \sqrt{10} \qquad (1)$$
$$y = 3x - 9 \qquad (2)$$

Starting from eqn (1):

$$(x-4)^2+(y-3)^2=10 \qquad (3)$$

(2) → (3):

$$(x-4)^2+(3x-9-3)^2=10$$
$$(x-4)^2+(3x-12)^2=10$$
$$x^2-8x+16+9x^2-72x+144=10$$
$$10x^2-80x+150=0$$
$$x^2-8x+15=0$$
$$(x-5)(x-3)=0$$

$$x=5 \text{ or } x=3$$

This gives two values of x that satisfy equations (1) and (2) Thus there are two points that will satisfy the conditions listed above. This is what we were expecting since B and D satisfy those conditions. So we can find the corresponding y values:

At $x=5$:

$$y=3(5)-9=6$$

At $x=3$:

$$y=3(3)-9=0$$

So the coordinates of B are $(3,0)$ and the coordinates of D are $(5,6)$.

[Note: The answer at the back of the text book is incorrect.]

Question 6

a) Rearrange the equation of l_1:

$$3x+4y=16$$
$$y=-\frac{3}{4}x+4$$

So the gradient of l_2 is $\frac{4}{3}$ and its equation is $y=\frac{4}{3}x+c$. At $(7,5)$:

$$5=\frac{4}{3}(7)+c$$
$$c=5-\frac{28}{3}$$
$$c=-\frac{13}{3}$$

So the equation for l_2 is $y=\frac{4}{3}x-\frac{13}{3}$.

b)

$$y=-\frac{3}{4}x+4 \qquad (1)$$
$$y=\frac{4}{3}x-\frac{13}{3} \qquad (2)$$

(2) → (1):

$$\frac{4}{3}x-\frac{13}{3}=-\frac{3}{4}x+4$$
$$16x-52=-9x+48$$
$$25x=100$$
$$x=4 \qquad (3)$$

(3) → (1):

$$y=-\frac{3}{4}(4)+4$$

$$y = 1 \qquad (4)$$

So the lines intersect at $(4,1)$.

c) The perpendicular distance of P from the line l_1 is the distance from P to the point of intersection:

$$d = \sqrt{(7-4)^2 + (5-1)^2}$$
$$= \sqrt{9+16}$$
$$= 5$$

Question 9

Rearrange the given equation:

$$2x + 7y = 5$$

$$y = -\frac{2}{7}x + \frac{5}{7}$$

So the gradient of the line is $-\frac{2}{7}$. The equation is $y = -\frac{2}{7}x + c$. At $(1,3)$:

$$3 = -\frac{2}{7}(1) + c$$

$$c = 3 + \frac{2}{7} = \frac{23}{7}$$

So the equation of the line is

$$y = -\frac{2}{7}x + \frac{23}{7}$$

$$or \quad 2x + 7y = 23$$

Question 10

First we find the gradient of the line joining $(2,-5)$ and $(-4,3)$:

$$gradient = \frac{3-(-5)}{-4-2} = \frac{8}{-6} = -\frac{4}{3}$$

So the gradient of the line we are looking for is $\frac{3}{4}$. The line passes through the midpoint of the line joining $(2,-5)$ and $(-4,3)$:

$$midpoint = \left(\frac{1}{2}(2+(-4)), \frac{1}{2}(-5+3)\right)$$
$$= (-1,-1)$$

The equation of the line is $y = \frac{3}{4}x + c$. At $(-1,-1)$:

$$-1 = \frac{3}{4}(-1) + c$$

$$c = -1 + \frac{3}{4} = -\frac{1}{4}$$

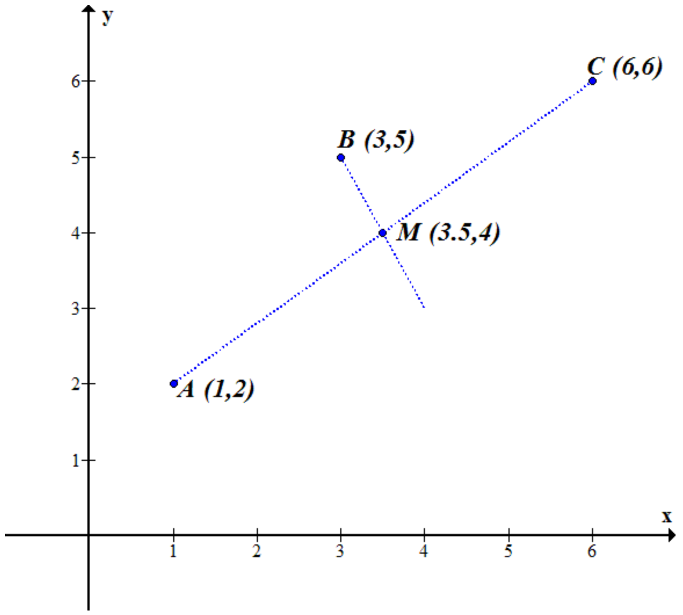
The equation of the line is $y = \frac{3}{4}x - \frac{1}{4}$.

Question 11

$$midpoint\ of\ AC = \left(\frac{1}{2}(1+6), \frac{1}{2}(2+6)\right)$$

$$= \left(3\frac{1}{2}, 4\right)$$

To find the coordinates of D, first plot the information on a graph:



The point D lies on the diagonal BD and is the same distance from M as B is from M. These conditions give two equations which can be solved to find the coordinates of D.

$$\begin{aligned} distance\ from\ B\ to\ M &= \sqrt{(3-3.5)^2+(5-4)^2} \\ &= \sqrt{0.25+1} \\ &= \sqrt{1.25} \end{aligned}$$

Equation of the diagonal BD:

$$gradient = \frac{4-5}{3.5-3} = -\frac{1}{0.5} = -2$$

The equation of the diagonal BD is $y = -2x + c$. At $(3,5)$:

$$5 = -2(3) + c$$

$$c = 11$$

Equation of BD:

$$y = -2x + 11$$

So the point $D\ (x, y)$ satisfies the following equations:

$$\sqrt{(x-3.5)^2+(y-4)^3} = \sqrt{1.25} \qquad (1)$$

$$y = -2x + 11 \qquad (2)$$

Starting with eqn (1):

$$(x-3.5)^2+(y-4)^2 = 1.25 \qquad (3)$$

(2) → (3):

$$\begin{aligned} (x-3.5)^2+(-2x+7)^2 &= 1.25 \\ x^2-7x+12.25+4x^2-28x+49 &= 1.25 \\ 5x^2-35x+60 &= 0 \\ x^2-7x+12 &= 0 \\ (x-3)(x-4) &= 0 \end{aligned}$$

$$x = 3\ or\ x = 4$$

The x coordinate of D is 4 since the x coordinate of B is 3. So

$$y = -2(4) + 11 = 3$$

The coordinates of D are $(4,3)$.

Question 12

a) The line AP is perpendicular to the line $y = 3x$. So the gradient of AP is $-\frac{1}{3}$. The equation of AP is $y = -\frac{1}{3}x + c$. At $(0,3)$:

$$3 = -\frac{1}{3}(0) + c$$

$$c = 3$$

So the equation of AP is $y = -\frac{1}{3}x + 3$.

b) The lines AP and $y = 3x$ intersect at the point P:

$$3x = -\frac{1}{3}x + 3$$

$$9x + x = 9$$

$$x = 0.9$$

Solve for y:

$$y = 3(0.9)$$

$$y = 2.7$$

The coordinates of P are $(0.9,2.7)$.

c) The perpendicular distance of A from the line $y = 3x$ is the distance between the point A and the point P:

$$distance = \sqrt{(0.9-0)^2 + (2.7-3)^2}$$

$$= \sqrt{0.9^2 + 0.3^2}$$

$$= \sqrt{0.9}$$

Question 13

To show that the given points are collinear, we must show that one of the points lies on the line passing through the other two points.

Equation of the line passing through $(-11,-5)$ and $(4,7)$:

$$gradient = \frac{7-(-5)}{4-(-11)} = \frac{12}{15} = \frac{4}{5}$$

Equation of line: $y = \frac{4}{5}x + c$. At $(4,7)$:

$$7 = \frac{4}{5}(4) + c$$

$$c = 7 - \frac{16}{5} = \frac{19}{5}$$

So the equation of the line passing through $(-11,-5)$ and $(4,7)$ is $y = \frac{4}{5}x + \frac{19}{5}$.

To see if this line passes through the point $(-1,3)$ find the value of y when $x = -1$:

$$y = \frac{4}{5}(-1) + \frac{19}{5}$$

$$= \frac{15}{5} = 3$$

So this line does indeed pass through the point $(-1,3)$. So the given points are collinear.

Question 14

i) To find the value of k substitute the coordinates of the point A into the equation of the line:

$$4x + ky = 20$$
$$4(8) + k(-4) = 20$$
$$32 - 4k = 20$$
$$4k = 12$$
$$k = 3$$

So the equation of the line is $4x + 3y = 20$.

Now we can find the value of b, by substituting $(b, 2b)$ into the equation of the line:

$$4(b) + 3(2b) = 20$$
$$10b = 20$$
$$b = 2$$

ii)

$$midpoint = \left(\frac{1}{2}(8+2), \frac{1}{2}(-4+4)\right) = (5,0)$$

[Note: The answer at the back of the text book is incorrect.]

Question 15

The point D is the point of intersection of the lines AD and DC. Since AD is perpendicular to AB and the coordinates of both A and B are given, we can find the equation of AD:

$$gradient\ of\ AB = \frac{8-2}{3-6} = \frac{6}{-3} = -2$$

So the gradient of AD is $\frac{1}{2}$. The equation of AD is $y = \frac{1}{2}x + c$. At $(3,8)$:

$$8 = \frac{1}{2}(3) + c$$
$$c = 8 - \frac{3}{2} = \frac{13}{2}$$

So the equation of AD is $y = \frac{1}{2}x + \frac{13}{2}$.

The line DC is parallel to AB, so its gradient is -2. The equation of DC is $y = -2x + c$. At $(10,2)$:

$$2 = -2(10) + c$$
$$c = 2 + 20 = 22$$

So the equation of DC is $y = -2x + 22$.

Now the point D is the intersection of these two lines. So we can equate the right hand sides of the two equations and solve for x:

$$\frac{1}{2}x + \frac{13}{2} = -2x + 22$$
$$x + 4x = 44 - 13$$
$$5x = 31$$
$$x = \frac{31}{5} = 6\frac{1}{5}$$
$$\therefore \quad y = -2\left(\frac{31}{5}\right) + 22$$
$$y = \frac{48}{5} = 9\frac{3}{5}$$

So the coordinates of D are $\left(6\frac{1}{5}, 9\frac{3}{5}\right)$.

[Note: The answer at the back of the text book is incorrect.]

Question 17

The point B is the intersection of the line segments AC and BD. The equation of the line segment BD can be obtained since it is perpendicular to AC and the coordinates of the point D are given.

Rearrange the equation of AC:

$$2y = x + 4 \qquad (1)$$

$$y = \frac{1}{2}x + 2$$

So the gradient of BD is -2. The equation of BD is $y = -2x + c$. At $(10, -3)$:

$$-3 = -2(10) + c$$

$$c = -3 + 20 = 17$$

So the equation of BD is

$$y = -2x + 17 \qquad (2)$$

Now find the point at which AC intersects BD:

(2) → (1):

$$2(-2x + 17) = x + 4$$

$$-4x - x = 4 - 34$$

$$-5x = -30$$

$$x = 6$$

$$\therefore \quad y = -2(6) + 17$$

$$y = 5$$

So the coordinates of B are $(6,5)$.

The point C lies on the line $2y = x + 4$ and is the same distance from the point B as the point A is from B. Thus, to find the coordinates of C we first need to find the coordinates of A and the distance from A to B.

The point A lies on the y-axis, so its x coordinate is zero. We can find the y coordinate by substituting $x = 0$ into eqn (1):

$$2y = 4$$

$$y = 2$$

So the coordinates of A are $(0,2)$.

$$distance\ from\ B\ to\ A = \sqrt{(6-0)^2 + (5-2)^2}$$

$$= \sqrt{36+9} = \sqrt{45}$$

So the point C (x, y) satisfies the following equations:

$$(x-6)^2 + (y-5)^2 = 45 \qquad (1)$$

$$y = \frac{1}{2}x + 2 \qquad (2)$$

(2) → (1):

$$(x-6)^2 + \left(\frac{1}{2}x - 3\right)^2 = 45$$

$$x^2 - 12x + 36 + \frac{1}{4}x^2 - 3x + 9 = 45$$

$$\frac{5}{4}x^2 - 15x = 0$$

$$x^2 - 12x = 0$$

$$x(x - 12) = 0$$

$$x = 0\ or\ x = 12$$

The x coordinate of C is 12, since the x coordinate of A is zero. So we can substitute this value of x into eqn (2) to solve for y:

$$y = \frac{1}{2}(12) + 2$$

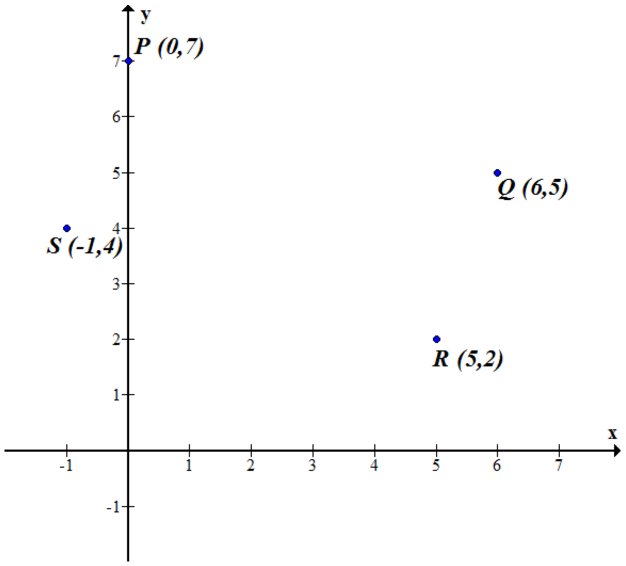
$$y = 6 + 2 = 8$$

So the coordinates of the point C are $(12,8)$.

[Note: The answer at the back of the text book is incorrect.]

Question 18

Plot the given points on a coordinate plane:



A rectangle has two sets of parallel lines, each set having the same length and all its corners are right angles. To show that the given points satisfy these conditions we must show that the distance between S and P is the same as the distance between R and Q and that the distance between R and S is the same the distance between P and Q. We must also show that RS is parallel to QP and that SP is parallel to RQ and that SP and RQ are perpendicular to RS and PQ. We start by finding the distances between the points:

$$distance\ from\ S\ to\ P = \sqrt{(-1-0)^2 + (4-7)^2} = \sqrt{10}$$

$$distance\ from\ R\ to\ Q = \sqrt{(5-6)^2 + (2-5)^2} = \sqrt{10}$$

$$distance\ from\ S\ to\ R = \sqrt{(-1-5)^2 + (4-2)^2} = \sqrt{40}$$

$$distance\ from\ P\ to\ Q = \sqrt{(0-6)^2 + (7-5)^2} = \sqrt{40}$$

So we see that

$$distance\ from\ S\ to\ P = distance\ from\ R\ to\ Q$$

$$distance\ from\ S\ to\ R = distance\ from\ P\ to\ Q$$

Next we find the gradients of the line segments joining the points:

$$gradient\ of\ SP = \frac{4-7}{-1-0} = 3$$

$$gradient\ of\ RP = \frac{5-2}{6-5} = 3$$

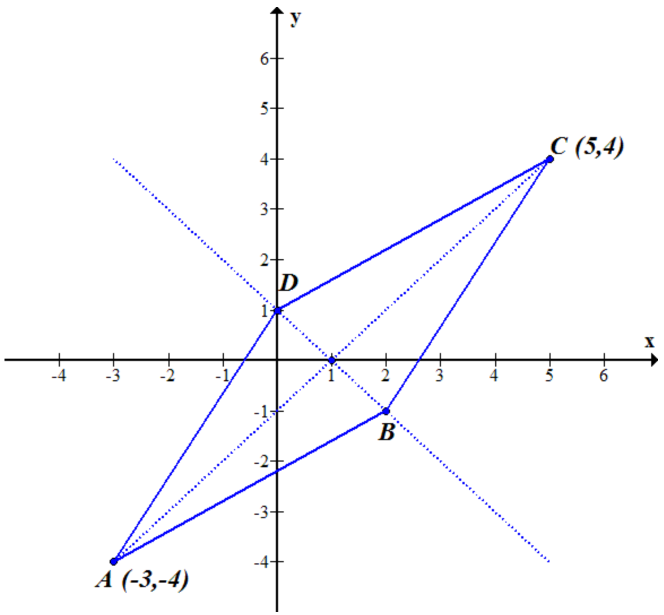
$$gradient\ of\ SR = \frac{4-2}{-1-5} = -\frac{1}{3}$$

$$gradient\ of\ PQ = \frac{7-5}{0-6} = -\frac{1}{3}$$

So it is clear that SP and RP are parallel and SR and PQ are parallel. It is also clear that SP and RP are perpendicular to SR and PQ. So the parallel line segments have the same length and all the corners are right angled. Thus these points do form a rectangle.

Question 20

a) The diagonals of a rhombus bisect each other at $90°$, but unlike a square they are not the same length. Start by plotting the points A and C and completing a rhombus using arbitrary points for B and D. Note: at this stage we are not certain whether or not D lies on the y-axis.



Since we are given the points A and C we can find the midpoint of AC and then the equation of the diagonal BD:

$$midpoint\ of\ AC = \left(\frac{1}{2}(-3+5), \frac{1}{2}(-4+4)\right) = (1,0)$$

$$gradient\ of\ AC = \frac{-4-4}{-3-5} = 1$$

BD is perpendicular to AC so the gradient of BD is -1. The equation of BD is $y = -x + c$. At $(1,0)$:

$$0 = -1(1) + c$$
$$c = 1$$

So the equation of the diagonal BD is $y = -x + 1$.

b) The point B is the intersection of the lines BD and BC. We are given the gradient of BC and the coordinates of C so we can find the equation of BC. The equation is $y = \frac{5}{3}x + c$. At $(5,4)$:

$$4 = \frac{5}{3}(5) + c$$
$$c = 4 - \frac{25}{3} = -\frac{13}{3}$$

So the equation of BC is $y = \frac{5}{3}x - \frac{13}{3}$.

Next we find the point of intersection of BD and BC:

$$-x + 1 = \frac{5}{3}x - \frac{13}{3}$$
$$5x + 3x = 3 + 13$$
$$8x = 16$$
$$x = 2$$
$$\therefore \quad y = -2 + 1 = -1$$

The coordinates of B are $(2,-1)$.

The point D is the intersection of the line segments BD and AD. Since AD is parallel to BC we can find the equation of AD. It is given by $y=\frac{5}{3}x+c$. At $A\ (-3,-4)$:

$$-4=\frac{5}{3}(-3)+c$$

$$c=-4+5=1$$

So the equation of AD is $y=\frac{5}{3}x+1$.

Now we find the point of intersection between AD and BC:

$$-x+1=\frac{5}{3}x+1$$

$$3x+5x=0$$

$$x=0$$

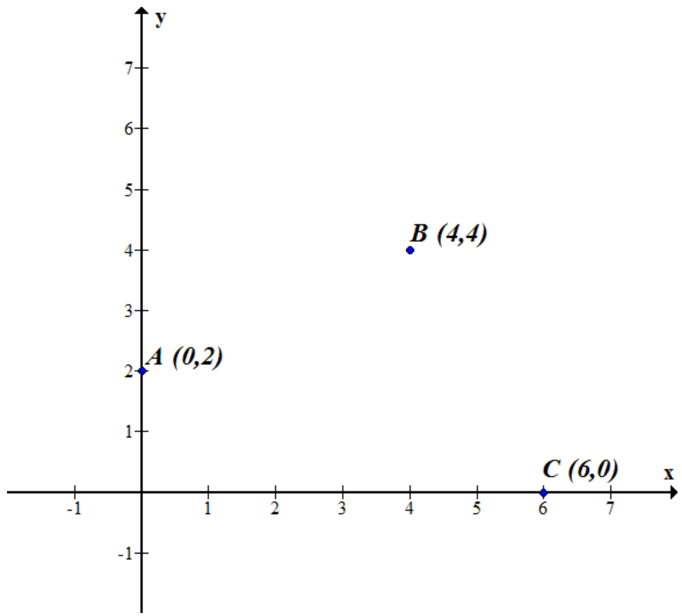
$$\therefore \quad y=-(0)+1=1$$

The coordinates of D are $(0,1)$.

Question 21

Note: a median is the line joining a vertex of a triangle to the midpoint of the opposite side.

First plot and label the points:



Next we find the midpoints of each side:

$$M_1=midpoint\ of\ AB=\left(\frac{1}{2}(0+4),\frac{1}{2}(2+4)\right)=(2,3)$$

$$M_2=midpoint\ of\ BC=\left(\frac{1}{2}(4+6),\frac{1}{2}(4+0)\right)=(5,2)$$

$$M_3=midpoint\ of\ AC=\left(\frac{1}{2}(0+6),\frac{1}{2}(2+0)\right)=(3,1)$$

Now find the equation of each median:

$$gradient\ of\ AM_2=\frac{2-2}{0-5}=0$$

The equation of AM_2 is $y = c$. Since the line passes through $(0,2)$, the value of c is 2. So the equation of AM_2 is

$$y = 2 \qquad (1)$$

$$gradient\ of\ BM_3 = \frac{4-1}{4-3} = 3$$

The equation of BM_3 is $y = 3x + c$. At $(4,4)$:

$$4 = 3(4) + c$$
$$c = 4 - 12 = -8$$

So the equation of BM_3 is

$$y = 3x - 8 \qquad (2)$$

$$gradient\ of\ CM_1 = \frac{0-3}{6-2} = -\frac{3}{4}$$

The equation of CM_1 is $y = -\frac{3}{4}x + c$. At $(6,0)$:

$$0 = -\frac{3}{4}(6) + c$$
$$c = \frac{9}{2}$$

So the equation of CM_1 is

$$y = -\frac{3}{4}x + \frac{9}{2} \qquad (3)$$

To show that all the medians are concurrent, we find the point of intersection of AM_2 and BM_3 and show that CM_1 passes through this point:

(1) → (2):

$$2 = 3x - 8$$
$$3x = 10$$
$$x = \frac{10}{3}$$

So AM_2 and BM_3 intersect at $\left(\frac{10}{3}, 2\right)$. To show that CM_1 passes through $\left(\frac{10}{3}, 2\right)$, we find the y value of CM_1 when $x = \frac{10}{3}$.

$$y = -\frac{3}{4}\left(\frac{10}{3}\right) + \frac{9}{2} = \frac{4}{2} = 2$$

So the medians are concurrent and all pass through the point $(\frac{10}{3}, 2)$.

CHAPTER 2

EXERCISE 2A

Question 1

a) $\sqrt{3} \times \sqrt{3} = \sqrt{3 \times 3} = 3$

c) $\sqrt{16} \times \sqrt{16} = \sqrt{16 \times 16} = 16$

e) $\sqrt{32} \times \sqrt{2} = \sqrt{32 \times 2} = \sqrt{64} = 8$

g) $5\sqrt{3} \times \sqrt{3} = 5\sqrt{3 \times 3} = 5(3) = 15$

i) $3\sqrt{6} \times 4\sqrt{6} = 12\sqrt{6 \times 6} = 12(6) = 72$

k) $\left(2\sqrt{7}\right)^2 = 2^2\left(\sqrt{7}\right)^2 = 4(7) = 28$

m) $\sqrt[3]{5} \times \sqrt[3]{5} \times \sqrt[3]{5} = \sqrt[3]{5 \times 5 \times 5} = 5$

o) $\left(2\sqrt[3]{2}\right)^6 = 2^6\left(\sqrt[3]{2}\right)^6 = 64(2)^2 = 256$

Question 2

a) $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$

c) $\sqrt{24} = \sqrt{4 \times 6} = \sqrt{4} \times \sqrt{6} = 2\sqrt{6}$

e) $\sqrt{40} = \sqrt{4 \times 10} = \sqrt{4} \times \sqrt{10} = 2\sqrt{10}$

g) $\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = 4\sqrt{3}$

i) $\sqrt{54} = \sqrt{9 \times 6} = \sqrt{9} \times \sqrt{6} = 3\sqrt{6}$

k) $\sqrt{135} = \sqrt{9 \times 15} = \sqrt{9} \times \sqrt{15} = 3\sqrt{15}$

Question 3

a) $\sqrt{8} + \sqrt{18} = \sqrt{4 \times 2} + \sqrt{9 \times 2} = 2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$

c) $\sqrt{20} - \sqrt{5} = \sqrt{4 \times 5} - \sqrt{5} = 2\sqrt{5} - \sqrt{5} = \sqrt{5}$

d) $\sqrt{32} - \sqrt{8} = \sqrt{16 \times 2} - \sqrt{4 \times 2} = 4\sqrt{2} - 2\sqrt{2} = 2\sqrt{2}$

f) $\sqrt{27} + \sqrt{27} = 2\sqrt{27} = 2\sqrt{9 \times 3} = 2\left(3\sqrt{3}\right) = 6\sqrt{3}$

g)

$$\begin{aligned}\sqrt{99} + \sqrt{44} + \sqrt{11} &= \sqrt{9 \times 11} + \sqrt{4 \times 11} + \sqrt{11} \\ &= 3\sqrt{11} + 2\sqrt{11} + \sqrt{11} \\ &= 6\sqrt{11}\end{aligned}$$

i) $2\sqrt{20} + 3\sqrt{45} = 2\sqrt{4 \times 5} + 3\sqrt{9 \times 5} = 4\sqrt{5} \times 9\sqrt{5} = 13\sqrt{5}$

j) $\sqrt{52} - \sqrt{13} = \sqrt{4 \times 13} - \sqrt{13} = 2\sqrt{13} - \sqrt{13} = \sqrt{13}$

l)

$$\begin{aligned}\sqrt{48} + \sqrt{24} - \sqrt{75} + \sqrt{96} &= \sqrt{16 \times 3} + \sqrt{4 \times 6} - \sqrt{25 \times 3} + \sqrt{16 \times 6} \\ &= 4\sqrt{3} + 2\sqrt{6} - 5\sqrt{3} + 4\sqrt{6} \\ &= 6\sqrt{6} - \sqrt{3}\end{aligned}$$

Question 4

a) $\frac{\sqrt{8}}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}} = 2$

c) $\frac{\sqrt{40}}{\sqrt{10}} = \frac{2\sqrt{10}}{\sqrt{10}} = 2$

e) $\frac{\sqrt{125}}{\sqrt{5}} = \frac{5\sqrt{5}}{\sqrt{5}} = 5$

g) $\frac{\sqrt{3}}{\sqrt{48}} = \frac{\sqrt{3}}{4\sqrt{3}} = \frac{1}{4}$

Question 5

a) $\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$

c) $\frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$

e) $\frac{11}{\sqrt{11}} = \frac{11}{\sqrt{11}} \times \frac{\sqrt{11}}{\sqrt{11}} = \frac{11\sqrt{11}}{11} = \sqrt{11}$

g) $\frac{12}{\sqrt{3}} = \frac{12}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$

i) $\frac{\sqrt{6}}{\sqrt{2}} = \frac{\sqrt{6}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{12}}{2} = \frac{2\sqrt{3}}{2} = \sqrt{3}$

k) $\frac{3\sqrt{5}}{\sqrt{3}} = \frac{3\sqrt{5}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{15}}{3} = \sqrt{15}$

m) $\frac{7\sqrt{2}}{2\sqrt{3}} = \frac{7\sqrt{2}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{7\sqrt{6}}{2(3)} = \frac{7}{6}\sqrt{6}$

o) $\frac{9\sqrt{12}}{2\sqrt{18}} = \frac{9(2)\sqrt{3}}{2(3)\sqrt{2}} = \frac{18\sqrt{3}}{6\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{18\sqrt{6}}{6(2)} = \frac{3}{2}\sqrt{6}$

Question 6

a) $\sqrt{75} + \sqrt{12} = 5\sqrt{3} + 2\sqrt{3} = 7\sqrt{3}$

c) $\frac{12}{\sqrt{3}} - \sqrt{27} = \frac{12\sqrt{3}}{3} - 3\sqrt{3} = 4\sqrt{3} - 3\sqrt{3} = \sqrt{3}$

d) $\frac{2}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{6}} = \frac{2\sqrt{3}}{3} + \frac{\sqrt{12}}{6} = \frac{2\sqrt{3}}{3} + \frac{2\sqrt{3}}{6} = \frac{(2+1)\sqrt{3}}{3} = \sqrt{3}$

f)

$$\left(3-\sqrt{3}\right)\left(2-\sqrt{3}\right) - \sqrt{3} \times \sqrt{27} = \left(6 - 3\sqrt{3} - 2\sqrt{3} + 3\right) - \sqrt{3} \times 3\sqrt{3}$$

$$= \left(9 - 5\sqrt{3}\right) - 3(3)$$

$$= 9 - 9 - 5\sqrt{3}$$

$$= -5\sqrt{3}$$

Question 7

a)

$$area = 4\sqrt{5} \times \sqrt{10}$$

$$= 4\sqrt{5 \times 10} = 4\sqrt{50}$$

$$= 4\sqrt{25 \times 2} = 20\sqrt{2}$$

b) The diagonal AC is the hypotenuse of the right angled triangle ABC. We use Pythagoras's theorem to find its length:

$$AC^2 = AB^2 + BC^2$$

$$= \left(4\sqrt{5}\right)^2 + \left(\sqrt{10}\right)^2$$

$$= 16(5) + 10$$

$$= 90$$

$$\therefore \ AC = \sqrt{90} = 3\sqrt{10}$$

Question 8

a)

$$x\sqrt{2} = 10$$

$$x = \frac{10}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$

c)

$$z\sqrt{32} - 16 = z\sqrt{8} - 4$$
$$4z\sqrt{2} - 2z\sqrt{2} = 12$$
$$2z\sqrt{2} = 12$$
$$z = \frac{6}{\sqrt{2}} = 3\sqrt{2}$$

Question 9

a) $\sqrt[3]{24} = \sqrt[3]{8 \times 3} = 2\sqrt[3]{3}$

d) $\sqrt[3]{3000} - \sqrt[3]{375} = \sqrt[3]{1000 \times 3} - \sqrt[3]{125 \times 3} = 10\sqrt[3]{3} - 5\sqrt[3]{3} = 5\sqrt[3]{3}$

Question 10

Since each of the triangles is right-angled, we can use Pythagoras's theorem. Let's call the length of the unknown side x in each case.

a)

$$x^2 = 12^2 + 8^2$$
$$= 144 + 64$$
$$= 208$$
$$\therefore \quad x = \sqrt{208} = 4\sqrt{13}\ cm$$

b)

$$x = \left(10\sqrt{2}\right)^2 + \left(5\sqrt{3}\right)^2$$
$$= 100(2) + 25(3)$$
$$= 200 + 75$$
$$= 275$$
$$\therefore \quad x = \sqrt{275} = 5\sqrt{11}\ cm$$

d)

$$\left(3\sqrt{7}\right)^2 = x^2 + \left(3\sqrt{2}\right)^2$$
$$9(7) = x^2 + 9(2)$$
$$x^2 = 45$$
$$\therefore \quad x = 3\sqrt{5}\ cm$$

Question 11

To use the information that we are given, we have to first write each expression in the form $k\sqrt{26}$.

a)

$$\sqrt{104} = \sqrt{4 \times 26} = 2\sqrt{26} = 2(5.099019513593)$$
$$= 10.198\,039\,027\,2$$

b)

$$\sqrt{650} = \sqrt{25 \times 26} = 5\sqrt{26}$$
$$= 5(5.099019513593)$$
$$= 25.495\,097\,568\,0$$

c)

$$\frac{13}{\sqrt{26}} = \frac{13\sqrt{26}}{26} = \frac{1}{2}\sqrt{26} = \frac{1}{2}(5.099019513893)$$
$$= 2.549\,509\,756\,8$$

Question 12

$$7x - (3\sqrt{5})y = 9\sqrt{5} \qquad (1)$$

$$(2\sqrt{5})x + y = 34 \qquad (2)$$

Rearrange eqn (2):

$$y = 34 - (2\sqrt{5})x \qquad (3)$$

(3) → (1):

$$\begin{aligned} 7x - (3\sqrt{5})\left(34 - (2\sqrt{5})x\right) &= 9\sqrt{5} \\ 7x - 102\sqrt{5} + 6(5)x &= 9\sqrt{5} \\ 37x &= 111\sqrt{5} \\ x &= 3\sqrt{5} \qquad (4) \end{aligned}$$

(4) → (3):

$$\begin{aligned} y &= 34 - (2\sqrt{5})(3\sqrt{5}) \\ &= 34 - 6(5) \\ &= 4 \qquad (5) \end{aligned}$$

So the solution is $x = 3\sqrt{5}$ and $y = 4$.

Question 13

a)

$$\begin{aligned} (\sqrt{2} - 1)(\sqrt{2} + 1) &= 2 - \sqrt{2} + \sqrt{2} - 1 \\ &= 2 - 1 = 1 \end{aligned}$$

c)

$$\begin{aligned} (\sqrt{7} + \sqrt{3})(\sqrt{7} - \sqrt{3}) &= 7 - \sqrt{7}\sqrt{3} + \sqrt{3}\sqrt{7} - 3 \\ &= 7 - 3 = 4 \end{aligned}$$

e)

$$\begin{aligned} (4\sqrt{3} - \sqrt{2})(4\sqrt{3} + \sqrt{2}) &= 16(3) + 4\sqrt{3}\sqrt{2} - 4\sqrt{2}\sqrt{3} - 2 \\ &= 48 - 2 = 46 \end{aligned}$$

g)

$$\begin{aligned} (4\sqrt{7} - \sqrt{5})(4\sqrt{7} + \sqrt{5}) &= 16(7) + 4\sqrt{7}\sqrt{5} - 4\sqrt{5}\sqrt{7} - 5 \\ &= 112 - 5 = 107 \end{aligned}$$

Question 14

All the parts in question 13 show the same pattern:

$$(a + b)(a - b) = a^2 - b^2$$

We use the pattern that we found in question 13 to answer this question. (This is a very useful pattern and is called the difference of two squares.)

a) $(\sqrt{3} - 1)(\sqrt{3} + 1) = 3 - 1 = 2$

c) $(\sqrt{6} - \sqrt{2})(\sqrt{6} + \sqrt{2}) = 6 - 2 = 4$

e) $(\sqrt{11} + \sqrt{10})(\sqrt{11} - \sqrt{10}) = 11 - 10 = 1$

Question 15

a) For any real number, a, $a \times 1 = a$. Since any real number divided by itself is equal to 1, we have $\frac{\sqrt{3}+1}{\sqrt{3}+1} = 1$. Therefore $\frac{1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{1}{\sqrt{3}-1}$.

$$LHS = \frac{1}{\sqrt{3}-1} = \frac{1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{\sqrt{3}+1}{3-1} = \frac{\sqrt{3}+1}{2} = RHS$$

b)

$$LHS = \frac{1}{2\sqrt{2}+\sqrt{3}} \times \frac{2\sqrt{2}-\sqrt{3}}{2\sqrt{2}-\sqrt{3}} = \frac{2\sqrt{2}-\sqrt{3}}{4(2)-3} = \frac{2\sqrt{2}-\sqrt{3}}{5} = RHS$$

(Notice that in choosing what to multiply the given expression by, we used the difference of two squares so that the denominator would be rational.)

Question 16

a)

$$\frac{1}{2-\sqrt{3}} = \frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}$$

b)

$$\frac{1}{3\sqrt{5}-5} = \frac{1}{3\sqrt{5}-5} \times \frac{3\sqrt{5}+5}{3\sqrt{5}+5} = \frac{3\sqrt{5}+5}{9(5)-25} = \frac{3\sqrt{5}+5}{20}$$

c)

$$\frac{4\sqrt{3}}{2\sqrt{6}+3\sqrt{2}} = \frac{4\sqrt{3}}{2\sqrt{6}+3\sqrt{2}} \times \frac{2\sqrt{6}-3\sqrt{2}}{2\sqrt{6}-3\sqrt{2}}$$

$$= \frac{8\sqrt{18}-12\sqrt{6}}{4(6)-9(2)}$$

$$= \frac{24\sqrt{2}-12\sqrt{6}}{6}$$

$$= 4\sqrt{2}-2\sqrt{6}$$

EXERCISE 2B

Question 1

a) $a^2 \times a^3 \times a^7 = a^{2+3+7} = a^{12}$

c) $c^7 \div c^3 = c^{7-3} = c^4$

d) $d^5 \times d^4 = d^{5+4} = d^9$

f) $(x^3y^2)^2 = (x^3)^2(y^2)^2 = x^{3\times 2}y^{2\times 2} = x^6y^4$

g) $5g^5 \times 3g^3 = 15g^{5+3} = 15g^8$

i) $(2a^2)^3 \times (3a)^2 = (8a^{2\times 3})(9a^2) = 72a^6a^2 = 72a^8$

j) $(p^2q^3)^2 \times (pq^3)^3 = p^{2\times 2}q^{3\times 2}p^3q^{3\times 3} = p^4q^6p^3q^9 = p^7q^{15}$

l) $(6ac^3)^2 \div (9a^2c^5) = 36a^2c^6 \div 9a^2c^5 = 4a^{2-2}c^{6-5} = 4c$

m) $(3m^4n^2)^3 \times (2mn^2)^2 = 27m^{12}n^6 2 \times 4m^2n^4 = 108m^{14}n^{10}$

o) $(2xy^2z^3)^2 \div (2xy^2z^3) = 4x^2y^4z^6 \div 2xy^2z^3 = 2xy^2z^3$

Question 2

a) $2^{11} \times (2^5)^3 = 2^{11} \times 2^{15} = 2^{11+15} = 2^{26}$

c) $4^3 = (2^2)^3 = 2^{2\times 3} = 2^6$

e) $\frac{2^7 \times 2^8}{2^{13}} = \frac{2^{15}}{2^{13}} = 2^2$

g) $4^2 \div 2^4 = (2^2)^2 \div 2^4 = 2^4 \div 2^4 = 2^{4-4} = 2^0$

Question 3

a) $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

c) $5^{-1} = \frac{1}{5}$

e) $10^{-4} = \frac{1}{10^4} = \frac{1}{10\,000}$

g) $\left(\frac{1}{2}\right)^{-1} = \frac{1}{2^{-1}} = 2$

i) $\left(2\frac{1}{2}\right)^{-1} = \left(\frac{5}{2}\right)^{-1} = \frac{2}{5}$

k) $6^{-3} = \frac{1}{6^3} = \frac{1}{216}$

Question 4

a) $4x^{-3} = \frac{4}{x^3} = \frac{4}{2^3} = \frac{4}{8} = \frac{1}{2}$

c) $\frac{1}{4}x^{-3} = \frac{1}{4x^3} = \frac{1}{4(2^3)} = \frac{1}{4(8)} = \frac{1}{32}$

d) $\left(\frac{1}{4}x\right)^{-3} = \left(\frac{1}{4}\right)^{-3}x^{-3} = \frac{4^3}{x^3} = \frac{64}{2^3} = \frac{64}{8} = 8$

f) $(x \div 4)^{-3} = \left(\frac{x}{4}\right)^{-3} = \frac{4^3}{x^3} = \frac{64}{2^3} = \frac{64}{8} = 8$

Question 5

a) $(2y)^{-1} = \frac{1}{2y} = \frac{1}{2(5)} = \frac{1}{10}$

c) $\left(\frac{1}{2}y\right)^{-1} = \left(\frac{1}{2}\right)^{-1}y^{-1} = \frac{2}{y} = \frac{2}{5}$

d) $\frac{1}{2}y^{-1} = \frac{1}{2y} = \frac{1}{2(5)} = \frac{1}{10}$

f) $\frac{2}{(y^{-1})^{-1}} = \frac{2}{y^{-1\times -1}} = \frac{2}{y} = \frac{2}{5}$

Question 6

a) $a^4 \times a^{-3} = a^{4-3} = a$

c) $(c^{-2})^3 = c^{-2\times 3} = c^{-6} = \frac{1}{c^6}$

d) $d^{-1} \times 2d = \frac{2d}{d} = 2$

f) $\frac{f^{-2}}{f^3} = \frac{1}{f^2 f^3} = \frac{1}{f^5}$

g) $12g^3 \times (2g^2)^{-2} = 12g^3 \times 2^{-2}g^{-4} = \frac{12g^3}{4g^4} = \frac{3}{g}$

i) $(3i^{-2})^{-2} = 3^{-2}i^4 = \frac{1}{9}i^4$

j) $\left(\frac{1}{2}j^{-2}\right)^{-3} = \left(\frac{1}{2}\right)^{-3} j^6 = 2^3 j^6 = 8j^6$

l) $(p^2q^4r^3)^{-4} = (p^{2\times -4}q^{4\times -4}r^{3\times -4}) = p^{-8}q^{-16}r^{-12}$

m) $(4m^2)^{-1} \times 8m^3 = \frac{8m^3}{4m^2} = 2m$

o) $(2xy^2)^{-1} \times (4xy)^2 = \frac{16x^2y^2}{2xy^2} = 8x$

p) $(5a^3c^{-1})^2 \div (2a^{-1}c^2) = \frac{25a^6c^{-2}}{2a^{-1}c^2} = \frac{25a^6a}{2c^2c^2} = \frac{25a^7}{2c^4}$

r) $(3x^{-2}y)^2 \div (4xy)^{-2} = (9x^{-4}y^2)(4^2x^2y^2) = 9(16)x^{-2}y^4 = \frac{144y^4}{x^2}$

Question 7

a) We can rewrite the given equation so that the right-hand side is expressed in powers of 3:

$$3^x = 3^{-2}$$

So $x = -2$.

c) Rewriting the given equation:

$$2^z \times 2^{z-3} = 2^5$$

$$2^{z+z-3} = 2^5$$

So we have:

$$2z - 3 = 5$$
$$z = 4$$

d) Rewriting the given equation:

$$7^{3x-x+2} = 7^{-2}$$

So we have

$$2x + 2 = -2$$
$$x = -2$$

f) Rewriting the given equation:

$$3^t \times (3^2)^{t+3} = (3^3)^2$$
$$3^{t+2t+6} = 3^6$$

So we have:

$$3t + 6 = 6$$
$$t = 0$$

Question 8

a)

$$\begin{aligned} volume &= (3 \times 10^{-2}\ m)^3 \\ &= 3^3 \times 10^{-6}\ m^3 \\ &= 27 \times 10^{-6}\ m^3 \\ &= 2.7 \times 10^1 \times 10^{-6}\ m^3 \\ &= 2.7 \times 10^{-5}\ m^3 \end{aligned}$$

b)

$$surface\ area = 6 \times (3 \times 10^{-2}\ m)^2$$
$$= 6 \times (9 \times 10^{-4}\ m^2)$$
$$= 54 \times 10^{-4}\ m^2$$
$$= 5.4 \times 10^{-3}\ m^2$$

Question 9

$$speed = \frac{2 \times 10^{-1}\ km}{7.5 \times 10^{-3}\ h}$$
$$= 0.2667 \times 10^{-1} \times 10^{3}\ km\ h^{-1}$$
$$= (2.667 \times 10^{-1}) \times 10^{2}\ km\ h^{-1}$$
$$= 2.67 \times 10^{1}\ km\ h^{-1}$$
$$= 26.7\ km\ h^{-1}$$

Question 10

a)

$$V = \pi(2 \times 10^{-3}\ m)^2(80\ m)$$
$$= 80 \times 4 \times \pi \times 10^{-6}\ m^3$$
$$= 1005.3 \times 10^{-6}\ m^3$$
$$= 1.0 \times 10^{-3}\ m^3 \quad to\ 2\ s.f.$$

b)

$$8 \times 10^{-3}\ m^3 = \pi(5 \times 10^{-3}\ m)^2 l$$
$$l = \frac{8 \times 10^{-3}\ m^3}{\pi(25 \times 10^{-6}\ m^2)}$$
$$l = 0.1019 \times 10^{-3} \times 10^{6}\ m$$
$$l = 0.1019 \times 10^{3}\ m$$
$$l = 101.9\ m \quad to\ 1\ d.p.$$

c)

$$6 \times 10^{-3}\ m^3 = \pi r^2(61\ m)$$
$$r^2 = \frac{6 \times 10^{-3}\ m^3}{\pi(61\ m)}$$
$$r^2 = 0.03131 \times 10^{-3}\ m^2$$
$$r^2 = 3.131 \times 10^{-5}$$
$$r = \sqrt{3.131 \times 10^{-5}\ m^2}$$
$$r = 5.6 \times 10^{3}\ m \quad to\ 2\ s.f.$$

Question 11

a)

$$y = \frac{\lambda d}{a}$$
$$y = \frac{(7 \times 10^{-7})(5 \times 10^{-1})}{8 \times 10^{-4}}$$
$$y = \frac{35 \times 10^{-8}}{8 \times 10^{-4}}$$
$$y = 4.375 \times 10^{-4}$$

b)

$$\lambda = \frac{ay}{d}$$
$$\lambda = \frac{(2.7 \times 10^{-4})(10^{-3})}{0.6}$$

$$\lambda = \frac{2.7 \times 10^{-7}}{0.6}$$

$$\lambda = 4.5 \times 10^{-7}$$

Question 12

$$\frac{3^{5x+2}}{9^{1-x}} = \frac{27^{4+3x}}{729}$$

$$\frac{3^{5x+2}}{(3^2)^{1-x}} = \frac{(3^3)^{4+3x}}{729}$$

$$\frac{3^{5x+2}}{3^{2-2x}} = \frac{3^{12+9x}}{3^6}$$

$$3^{5x+2-2+2x} = 3^{12+9x-6}$$

$$3^{7x} = 3^{9x+6}$$

So we have:

$$7x = 9x + 6$$

$$2x = -6$$

$$x = -3$$

EXERCISE 2C

Question 1

a) $25^{\frac{1}{2}} = \sqrt{25} = 5$

c) $36^{\frac{1}{2}} = \sqrt{36} = 6$

e) $81^{\frac{1}{4}} = \sqrt[4]{81} = 3$

g) $16^{-\frac{1}{4}} = \frac{1}{16^{\frac{1}{4}}} = \frac{1}{\sqrt[4]{16}} = \frac{1}{2}$

k) $64^{\frac{2}{3}} = \left(\sqrt[3]{64}\right)^2 = 4^2 = 16$

l) $(-125)^{-\frac{4}{3}} = \left(\sqrt[3]{-125}\right)^{-4} = (-5)^{-4} = \frac{1}{(-5)^4} = \frac{1}{625}$

Question 2

a) $4^{\frac{1}{2}} = \sqrt{4} = 2$

c) $\left(\frac{1}{4}\right)^{-2} = 4^2 = 16$

e) $\left(\frac{1}{4}\right)^{\frac{1}{2}} = 4^{\frac{1}{2}} = 2$

h) $\left(\left(\frac{1}{4}\right)^{\frac{1}{4}}\right)^2 = \left(\frac{1}{4}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$

Question 3

a) $8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

c) $9^{-\frac{3}{2}} = \frac{1}{\left(\sqrt{9}\right)^3} = \frac{1}{3^3} = \frac{1}{27}$

e) $32^{\frac{2}{5}} = \left(\sqrt[5]{32}\right)^2 = 2^2 = 4$

g) $64^{-\frac{5}{6}} = \frac{1}{\left(\sqrt[6]{64}\right)^5} = \frac{1}{2^5} = \frac{1}{32}$

i) $10\,000^{-\frac{3}{4}} = \left(\sqrt[4]{10\,000}\right)^{-3} = 10^{-3} = \frac{1}{1000}$

k) $\left(3\frac{3}{8}\right)^{\frac{2}{3}} = \left(\frac{27}{8}\right)^{\frac{2}{3}} = \left(\frac{\sqrt[3]{27}}{\sqrt[3]{8}}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2\frac{1}{4}$

Question 4

a) $a^{\frac{1}{3}} \times a^{\frac{5}{3}} = a^{\frac{6}{3}} = a^2$

c) $\left(6c^{\frac{1}{4}}\right) \times (4c)^{\frac{1}{2}} = 6(2)c^{\frac{3}{4}} = 12c^{\frac{3}{4}}$

d) $(d^2)^{\frac{1}{3}} \div \left(d^{\frac{1}{3}}\right)^2 = d^{\frac{2}{3}} \div d^{\frac{2}{3}} = 1$

f) $(24e)^{\frac{1}{3}} \div (3e)^{\frac{1}{3}} = \frac{\sqrt[3]{24}e^{\frac{1}{3}}}{\sqrt[3]{3}e^{\frac{1}{3}}} = \sqrt[3]{\left(\frac{24}{3}\right)} = \sqrt[3]{8} = 2$

g) $\frac{(5p^2q^4)^{\frac{1}{3}}}{(25pq^2)^{-\frac{1}{3}}} = 5^{\frac{1}{3}}25^{\frac{1}{3}}p^{\frac{2}{3}}p^{\frac{1}{3}}q^{\frac{4}{3}}q^{\frac{2}{3}} = (125)^{\frac{1}{3}}pq^2 = 5pq^2$

i) $\frac{(2x^2y^{-1})^{-\frac{1}{4}}}{(8x^{-1}y^2)^{-\frac{1}{2}}} = 8^{\frac{1}{2}}2^{-\frac{1}{4}}x^{-\frac{1}{2}}y^{-1}x^{-\frac{1}{2}}y^{\frac{1}{4}} = (2^3)^{\frac{1}{2}}2^{-\frac{1}{4}}x^{-1}y^{-\frac{3}{4}} = 2^{\frac{5}{4}}x^{-1}y^{-\frac{3}{4}}$

Question 5

a)

$$x^{\frac{1}{2}} = 8$$

$$x = 8^2 = 64$$

c)

$$x^{\frac{2}{3}} = 4$$

$$x = 4^{\frac{3}{2}} = 8$$

e)

$$x^{-\frac{3}{2}} = 8$$

$$x = 8^{-\frac{2}{3}} = \frac{1}{4}$$

g)

$$x^{\frac{3}{2}} = x\sqrt{2}$$

$$x^{\frac{1}{2}} = 2^{\frac{1}{2}}$$

$$x = \left(2^{\frac{1}{2}}\right)^2 = 2$$

Question 6

a)

$$T = 2\pi l^{\frac{1}{2}}g^{-\frac{1}{2}}$$

$$T = 2\pi(0.9\ m)^{\frac{1}{2}}(9.81\ m\ s^{-2})^{-\frac{1}{2}}$$

$$T = 1.9\ s$$

b)

$$l^{\frac{1}{2}} = \frac{T}{2\pi g^{-\frac{1}{2}}}$$

$$l = \frac{T^2 g}{4\pi^2}$$

$$l = \frac{(3\ s)^2(9.81\ m\ s^{-2})}{4\pi^2}$$

$$l = 2.2\ m$$

Question 7

$$r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$$

$$r = \left(\frac{3(1150\ cm^3)}{4\pi}\right)^{\frac{1}{3}}$$

$$r = (274.5\ cm^3)^{\frac{1}{3}}$$

$$r = 6.5\ cm$$

Question 8

In each case we rewrite the given equation in the form $a^x = a^k$ where k is known and then solve for x.

a)

$$4^x = 32$$

$$2^{2x} = 2^5$$

$$\therefore\ 2x = 5$$

$$x = \frac{5}{2}$$

c)

$$16^z = 2$$

$$2^{4z} = 2$$

$$\therefore\ 4z = 1$$

$$z = \frac{1}{4}$$

e)

$$8^y = 16$$

$$2^{3y} = 2^4$$

$$\therefore\ 3y = 4$$

$$y = \frac{4}{3}$$

g)

$$(2^t)^3 \times 4^{t-1} = 16$$

$$2^{3t}2^{2t-2} = 2^4$$

$$2^{5t-2} = 2^4$$

$$\therefore\ 5t - 2 = 4$$

$$5t = 6$$

$$t = \frac{6}{5}$$

MISCELLANEOUS EXERCISE 2

Question 1

c)

$$\begin{aligned}\left(\sqrt{5}-2\right)^2+\left(\sqrt{5}-2\right)\left(\sqrt{5}+2\right)&=\left(5-2\sqrt{5}-2\sqrt{5}+4\right)+5-4\\&=10-4\sqrt{5}\end{aligned}$$

d)

$$\begin{aligned}\left(2\sqrt{2}\right)^5&=2^5 2^{\frac{5}{2}}\\&=2^5 2^{2\frac{1}{2}}\\&=2^{5+2}2^{\frac{1}{2}}\\&=2^7\sqrt{2}\\&=128\sqrt{2}\end{aligned}$$

Question 2

b) $\sqrt{63}-\sqrt{28}=\sqrt{9\times 7}-\sqrt{4\times 7}=3\sqrt{7}-2\sqrt{7}=\sqrt{7}$

d) $\sqrt[3]{2}+\sqrt[3]{16}=2^{\frac{1}{3}}+(2^4)^{\frac{1}{3}}=2^{\frac{1}{3}}+2^{\frac{4}{3}}=2^{\frac{1}{3}}+2^{\frac{1}{3}}2^{\frac{3}{3}}=(1+2)2^{\frac{1}{3}}=3\sqrt[3]{2}$

Question 3

a) $\frac{9}{2\sqrt{3}}=\frac{9}{2\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{9\sqrt{3}}{2(3)}=\frac{3}{2}\sqrt{3}$

c) $\frac{2\sqrt{5}}{3\sqrt{10}}=\frac{2\sqrt{5}}{3\sqrt{10}}\times\frac{\sqrt{10}}{\sqrt{10}}=\frac{2\sqrt{5}\sqrt{10}}{3(10)}=\frac{2\sqrt{5}\sqrt{5}\sqrt{2}}{30}=\frac{2(5)\sqrt{2}}{30}=\frac{10\sqrt{2}}{30}=\frac{1}{3}\sqrt{2}$

Question 4

c)

$$\begin{aligned}\frac{1}{\sqrt{2}}\left(2\sqrt{2}-1\right)+\sqrt{2}\left(1-\sqrt{8}\right)&=2-\frac{1}{\sqrt{2}}+\sqrt{2}-\sqrt{16}\\&=2-4-\frac{\sqrt{2}}{2}+\sqrt{2}\\&=-2+\frac{\sqrt{2}}{2}\end{aligned}$$

d)

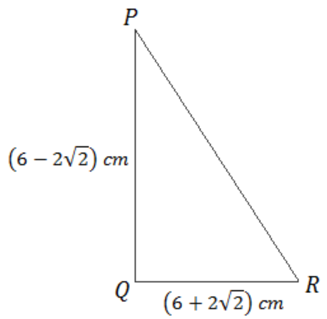
$$\begin{aligned}\frac{\sqrt{6}}{\sqrt{2}}+\frac{3}{\sqrt{3}}+\frac{\sqrt{15}}{\sqrt{5}}+\frac{\sqrt{18}}{\sqrt{6}}&=\frac{\sqrt{12}}{2}+\frac{3\sqrt{3}}{3}+\frac{\sqrt{75}}{5}+\frac{\sqrt{108}}{6}\\&=\frac{2\sqrt{3}}{2}+\sqrt{3}+\frac{5\sqrt{3}}{5}+\frac{6\sqrt{3}}{6}\\&=\sqrt{3}+\sqrt{3}+\sqrt{3}+\sqrt{3}\\&=4\sqrt{3}\end{aligned}$$

Question 6

a) $\sqrt{12}\times\sqrt{75}=2\sqrt{3}\times 5\sqrt{3}=10(3)=30$

b) $\sqrt{12}\times\sqrt{75}=(2^2\times 3)^{\frac{1}{2}}\times(5^2\times 3)^{\frac{1}{2}}=2\times 3^{\frac{1}{2}}\times 5\times 3^{\frac{1}{2}}=2\times 5\times 3=30$

Question 8



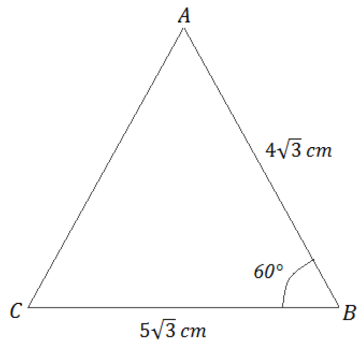
a)

$$area = \frac{1}{2} \times base \times height$$
$$area = \frac{1}{2}(6-2\sqrt{2})(6+2\sqrt{2})\ cm^2$$
$$area = \frac{1}{2}(36-4(2))\ cm^2$$
$$area = \frac{1}{2}(36-8)\ cm^2$$
$$area = 14\ cm^2$$

b) From the figure it is clear that the side PR is the hypotenuse of a right-triangle. Thus the length of PR can be found using Pythagoras's theorem:

$$PR^2 = (6-2\sqrt{2})^2 + (6+2\sqrt{2})^2$$
$$= 36 - 24\sqrt{2} + 4(2) + 36 + 24\sqrt{2} + 4(2)$$
$$= 88$$
$$\therefore \quad PR = \sqrt{88} = \sqrt{4 \times 22} = 2\sqrt{22}$$

Question 10



From the figure above it is clear that the side AC is opposite to the angle B. So the cosine rule gives us:

$$AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos(B)$$
$$AC^2 = (4\sqrt{3})^2 + (5\sqrt{3})^2 - 2(4\sqrt{3})(5\sqrt{3})\cos(60°)$$
$$AC^2 = 16(3) + 25(3) - 2(20)(3)\left(\frac{1}{2}\right)$$
$$AC^2 = 63$$
$$\therefore \quad AC = \sqrt{63} = \sqrt{9 \times 7} = 3\sqrt{7}$$

Question 11

$$5x - 3y = 41 \qquad (1)$$
$$(7\sqrt{2})x + (4\sqrt{2})y = 82 \qquad (2)$$

Rearrange eqn (1) to make y the subject:

$$3y = 5x - 41$$
$$y = \frac{5}{3}x - \frac{41}{3} \qquad (3)$$

(3) → (2):

$$(7\sqrt{2})x + (4\sqrt{2})\left(\frac{5}{3}x - \frac{41}{3}\right) = 82$$

$$(7\sqrt{2})x + \frac{20\sqrt{2}}{3}x - \frac{164\sqrt{2}}{3} = 82$$

$$\frac{41\sqrt{2}}{3}x = 82 + \frac{164\sqrt{2}}{3}$$

$$x = \frac{3(246 + 164\sqrt{2})}{41(3)\sqrt{2}}$$

$$x = \frac{6}{\sqrt{2}} + 4$$

$$x = 3\sqrt{2} + 4 \qquad (4)$$

(4) → (3):

$$y = \frac{5}{3}(3\sqrt{2} + 4) - \frac{41}{3}$$

$$y = 5\sqrt{2} + \frac{20}{3} - \frac{41}{3}$$

$$y = 5\sqrt{2} - 7 \qquad (5)$$

So the solution is $x = 3\sqrt{2} + 4$ and $y = 5\sqrt{3} - 7$.

Question 14

a) The equation of the line is $y = -\frac{1}{2}x + c$. At the point $A\ (2,3)$:

$$3 = -\frac{1}{2}(2) + c$$

$$c = 3 + 1 = 4$$

So the equation of the line l is $y = -\frac{1}{2}x + 4$.

b) The point $P\ (2 + 2t, 3 - t)$ lies on the line l if, for any value of t, it satisfies the equation found in part a. Starting from the right-hand side gives:

$$\begin{aligned} RHS &= -\frac{1}{2}(2 + 2t) + 4 \\ &= -1 - t + 4 \\ &= 3 - t \\ &= LHS \end{aligned}$$

c) The length of AP can be found using the distance formula:

$$AP = \sqrt{(2 + 2t - 2)^2 + (3 - t - 3)^2}$$

$$AP = \sqrt{(2t)^2 + (-t)^2}$$

$$AP = \sqrt{4t^2 + t^2}$$

$$AP = \sqrt{5t^2}$$

Now we want to find the values of t such that $AP = 5$:

$$\sqrt{5t^2} = 5$$

$$5t^2 = 25$$

$$t^2 = 5$$

$$t = \pm\sqrt{5}$$

So the length AP is 5 units if t is either $\sqrt{5}$ or $-\sqrt{5}$.

d) First we need to find an expression for the gradient of the line OP:

$$gradient\ OP = \frac{3-t-0}{2+2t-0} = \frac{3-t}{2+2t}$$

Now we are looking for the value of t such that OP is perpendicular to l, i.e. where the gradient of OP is 2.

$$\frac{3-t}{2+2t} = 2$$

$$3-t = 4+4t$$

$$5t = -1$$

$$t = -\frac{1}{5}$$

So the length of the perpendicular from O to l is the length of the lines segment OP, when $t = -\frac{1}{5}$:

$$length = \sqrt{\left(2+2\left(-\frac{1}{5}\right)\right)^2 + \left(3-\left(-\frac{1}{5}\right)\right)^2}$$

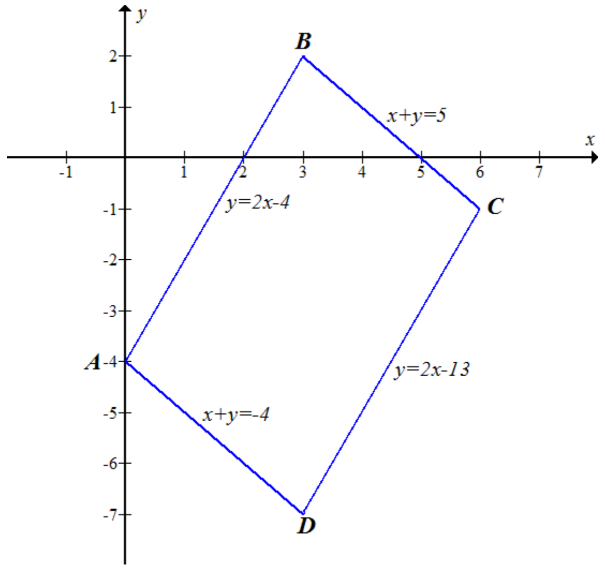
$$= \sqrt{\left(\frac{8}{5}\right)^2 + \left(\frac{16}{5}\right)^2}$$

$$= \sqrt{\left(\frac{64}{25}\right) + \left(\frac{256}{25}\right)}$$

$$= \sqrt{\frac{320}{25}} = \frac{8}{5}\sqrt{5}$$

Question 16

We start by plotting the parallelogram:



To find the length of the side AD, we need to find the coordinates of the points A and D. We do this by finding the point of intersection of the sides AD and CD and the point of intersection of the sides AB and AD:

$$x + y = -4 \qquad (1)$$

$$y = 2x - 4 \qquad (2)$$

(2) → (1):

$$x + 2x - 4 = -4$$

$$3x = 0$$

$$x = 0 \qquad (3)$$

(3) → (2):

$$y = 0 - 4 = -4 \tag{4}$$

So the coordinates of A are $(0, -4)$.

$$x + y = -4 \tag{5}$$

$$y = 2x - 13 \tag{6}$$

(6) → (5):

$$x + 2x - 13 = -4$$

$$3x = 9$$

$$x = 3 \tag{7}$$

(7) → (6):

$$y = 2(3) - 13 = -7 \tag{8}$$

So the coordinates of D are $(3, -7)$.

Now we can find the length of the side AD:

$$\begin{aligned} length &= \sqrt{(3-0)^2 + (-7+4)^2} \\ &= \sqrt{9+9} \\ &= \sqrt{18} = 3\sqrt{2} \end{aligned}$$

To find the perpendicular distance between the side AD and BC we find the equation of the line that is perpendicular to AD and passes through the point A. Let us call this line l. The equation of the line AD can be rewritten as

$$y = -x - 4.$$

So the gradient of l is 1. Equation of l: $y = x + c$. At $(0, -4)$:

$$-4 = 0 + c$$

$$c = -4$$

So the equation of the line l is $y = x - 4$.

Now we need to find the point at which the line l intersects the side BC:

$$x + y = 5 \tag{9}$$

$$y = x - 4 \tag{10}$$

(10) → (9):

$$x + x - 4 = 5$$

$$2x = 9$$

$$x = \frac{9}{2} \tag{11}$$

(11) → (10):

$$y = \frac{9}{2} - 4 = \frac{1}{2} \tag{12}$$

So the line l intersects the side BC at $\left(\frac{9}{2}, \frac{1}{2}\right)$. The perpendicular distance between sides AD and BC is the distance between the points $(0, -4)$ and $\left(\frac{9}{2}, \frac{1}{2}\right)$:

$$\begin{aligned} distance &= \sqrt{\left(\frac{9}{2} - 0\right)^2 + \left(\frac{1}{2} + 4\right)^2} \\ &= \sqrt{\left(\frac{9}{2}\right)^2 + \left(\frac{9}{2}\right)^2} \end{aligned}$$

$$= \sqrt{\frac{81}{4} + \frac{81}{4}}$$

$$= \sqrt{2\left(\frac{81}{4}\right)}$$

$$= \sqrt{\frac{81}{4}}\sqrt{2}$$

$$= \frac{9}{2}\sqrt{2}$$

So the area of the parallelogram is

$$area = 3\sqrt{2} \times \frac{9}{2}\sqrt{2} = \frac{27}{2}(2) = 27$$

Question 19

$$x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + 8 = 0$$

We cannot add the two terms involving x since they have different powers of x. However, if we make a substitution we can transform this equation into a quadratic equation which we can solve.

Let $y = x^{\frac{1}{3}}$. Then the equation becomes

$$y^2 - 6y + 8 = 0$$

$$(y-4)(y-2) = 0$$

$$y = 4 \ \ or \ \ y = 2$$

Now we must substitute back into our expression for y to solve for x:

$$x = y^3$$

$$x = 4^3 \ \ or \ \ x = 2^3$$

$$x = 64 \ \ or \ \ x = 8$$

[Note: the answer at the back of the text book is partially incorrect.]

Question 20

$$4^{2x} \times 8^{x-1} = 32$$

Rewrite this equation in the form $2^{ax+b} = 2^k$:

$$(2^2)^{2x} \times (2^3)^{x-1} = 2^5$$

$$2^{4x}2^{3x-3} = 2^5$$

$$2^{7x-3} = 2^5$$

So we have

$$7x - 3 = 5$$

$$7x = 8$$

$$x = \frac{8}{7}$$

Question 22

b)

$$\frac{(5b)^{-1}}{(8b^6)^{\frac{1}{3}}} = \frac{1}{5b(2b^2)} = \frac{1}{10b^3}$$

d)

$$\left(m^{\frac{1}{3}}n^{\frac{1}{2}}\right)^2 \times \left(m^{\frac{1}{6}}n^{\frac{1}{3}}\right)^4 \times (mn)^{-2} = m^{\frac{2}{3}}n^{\frac{2}{2}}m^{\frac{6}{4}}n^{\frac{4}{3}}m^{-2}n^{-2}$$

$$= m^{\frac{4}{3}}m^{-\frac{6}{3}}n^{\frac{4}{3}}n^{-1}$$

$$= m^{-\frac{2}{3}}n^{\frac{1}{3}}$$

Question 26

c) $2^{\frac{1}{3}} + 2^{\frac{1}{3}} + 2^{\frac{1}{3}} + 2^{\frac{1}{3}} = 4\left(2^{\frac{1}{3}}\right) = 2^2 2^{\frac{1}{3}} = 2^{\frac{7}{3}}$

e)

$$8^{0.1} + 8^{0.1} + 8^{0.1} + 8^{0.1} + 8^{0.1} + 8^{0.1} + 8^{0.1} + 8^{0.1} = 8(8^{0.1})$$

$$= 8^{1+1.1} = 8^{1.1} = (2^3)^{1.1} = 2^{3.3}$$

[Note: the answer to part (e) at the back of the text book is incorrect.]

Question 27

$$\frac{125^{3x}}{5^{x+4}} = \frac{25^{x-2}}{3125}$$

$$(5^3)^{3x}(5^{-x-4}) = \frac{(5^2)^{x-2}}{5^5}$$

$$5^{9x}5^{-x-4} = 5^{2x-4}5^{-5}$$

$$5^{8x-4} = 5^{2x-9}$$

So we have

$$8x - 4 = 2x - 9$$

$$6x = -5$$

$$x = -\frac{5}{6}$$

Question 28

a) Solve for r in the expression for V:

$$r^3 = \frac{3}{4}\frac{V}{\pi}$$

$$r = \left(\frac{3}{4}\frac{V}{\pi}\right)^{\frac{1}{3}}$$

Now substitute this into the expression for S:

$$S = 4\pi r^2$$

$$= 4\pi\left(\frac{3}{4}\frac{V}{\pi}\right)^{\frac{2}{3}}$$

$$= 2^2(2^2)^{-\frac{2}{3}}(3)^{\frac{2}{3}}\pi\pi^{-\frac{2}{3}}V^{\frac{2}{3}}$$

$$= 2^{\frac{2}{3}}3^{\frac{2}{3}}\pi^{\frac{1}{3}}V^{\frac{2}{3}}$$

[Note: the answer at the back of the text book is partially incorrect.]

b) Solve for r in the expression for S:

$$r^2 = \frac{S}{4\pi}$$

$$r = \left(\frac{S}{4\pi}\right)^{\frac{1}{2}}$$

Now substitute this into the expression for V:

$$V = \frac{4}{3}\pi r^3$$

$$= \frac{4}{3}\pi\left(\frac{S}{4\pi}\right)^{\frac{3}{2}}$$

$$= 4\left(4^{-\frac{3}{2}}\right)3^{-1}\pi\pi^{-\frac{3}{2}}S^{\frac{3}{2}}$$

$$= 4^{-\frac{1}{2}}3^{-1}\pi^{-\frac{1}{2}}S^{\frac{3}{2}}$$

$$= (2^2)^{-\frac{1}{2}}3^{-1}\pi^{-\frac{1}{2}}S^{\frac{3}{2}}$$

$$= 2^{-1}3^{-1}\pi^{-\frac{1}{2}}S^{\frac{3}{2}}$$

CHAPTER 3

EXERCISE 3A

Question 1

$f(x) = 2x + 5$

a) $f(3) = 2(3) + 5 = 11$

b) $f(0) = 2(0) + 5 = 5$

c) $f(-4) = 2(-4) + 5 = 3$

d) $f\left(-2\frac{1}{2}\right) = 2\left(-2\frac{1}{2}\right) + 5 = 2\left(-\frac{5}{2}\right) + 5 = 0$

Question 2

$f(x) = 3x^2 + 2$

a) $f(4) = 3(4)^2 + 2 = 3(16) + 2 = 50$

b) $f(1) = 3(1)^2 + 2 = 3 + 2 = 5$

$f(-1) = 3(-1)^2 + 2 = 3 + 2 = 5$

(Note that $(-x)^2 = x^2$ so the function has the same value at $-x$ or x.)

c) $f(3) = 3(3)^2 + 2 = 3(9) + 2 = 29$

$f(-3) = 3(-3)^2 + 2 = 3(9) + 2 = 29$

d) $f(3) = 3(3)^2 + 2 = 3(9) + 2 = 29$

Question 3

$f(x) = x^2 + 4x + 3$

a) $f(2) = (2)^2 + 4(2) + 3 = 4 + 8 + 3 = 15$

b) $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right) + 3 = \frac{1}{4} + 2 + 3 = 5\frac{1}{4}$

c) $f(1) = (1)^1 + 4(1) + 3 = 1 + 4 + 3 = 8$

$f(-1) = (-1)^2 + 4(-1) + 3 = 1 - 4 + 3 = 0$

(Note: this function has the term $4x$ and $4(x) \neq 4(-x)$ so the function has different values at x and $-x$.)

d) $f(3) = (3)^2 + 4(3) + 3 = 9 + 12 + 3 = 24$

$f(-3) = (-3)^2 + 4(-3) + 3 = 9 - 12 + 3 = 0$

Question 4

$g(x) = x^3$ and $h(x) = 4x + 1$

a) $g(2) + h(2) = (2)^3 + 4(2) + 1 = 8 + 8 + 1 = 17$

b)

$$\begin{aligned} 3g(-1) - 4h(-1) &= 3(-1)^3 - 4(4(-1) + 1) \\ &= -3 - 4(-4) - 4 \\ &= 9 \end{aligned}$$

c)

$$g(5) = 5^3 = 125$$

$$h(31) = 4(31) + 1 = 124 + 1 = 125$$

$$\therefore \quad g(5) = h(31)$$

d) $h\big(g(2)\big) = 4\big(g(2)\big) + 1 = 4(2^3) + 1 = 4(8) + 1 = 33$

Question 5

We are given $f(x) = x^n$ and $f(3) = 81$, so we substitute 3 into the function to determine the value of n:

$$f(3) = 3^n = 81 = 3^4$$

So $n = 4$.

Question 6

We are given that $f(x) = ax + b$ and that $f(2) = 7$and $f(3) = 12$, so we substitute 2 and 3 into the function and solve the two resulting equations simultaneously to obtain the values of a and b:

$$f(2) = 2a + b = 7 \qquad (1)$$

$$f(3) = 3a + b = 12 \qquad (2)$$

(1) – (2):

$$2a + b - 3a - b = 7 - 12$$

$$-a = -5$$

$$a = 5 \qquad (3)$$

(3) → (1):

$$2(5) + b = 7$$

$$b = 7 - 10 = -3 \qquad (4)$$

So the function is $f(x) = 5x - 3$.

Question 7

The largest possible domain of a function is the largest set of numbers for which the function is defined.

a) The function $\sqrt{x}$ is defined for all positive real numbers x and zero. To put it the other way around, the square root of a negative number is undefined. So the largest possible domain of the function is the set of all real numbers x such that $x \geq 0$.

e) Since the square root of a negative number is undefined, the largest possible domain of the function is the set of all real numbers x such that

$$x(x - 4) \geq 0$$

Now the product of two real numbers is positive if either both numbers are positive or both numbers are negative. So we have

$$x \geq 0 \; and \; x - 4 \geq 0 \; OR \; x \leq 0 \; and \; x - 4 \leq 0$$

$$x \geq 0 \; and \; x \geq 4 \; OR \; x \leq 0 \; and \; x \leq 4$$

Note that if $x \geq 0 \; and \; x \geq 4$, then x must be greater than or equal to 4. Similar reasoning applies to the statement after 'OR' above. Thus the solution of the above inequalities is

$$x \geq 4 \; or \; x \leq 0$$

So the largest domain of this function is the set of all real numbers x such that $x \leq 0 \; or \; x \geq 4$.

h) Again, we are looking for the set of values for x that always give a positive number under the square root:

$$x^3 - 8 \geq 0$$

$$x^3 \geq 8$$

$$x \geq 2$$

So the domain of the function is the set of all real numbers x such that $x \geq 2$.

i) Here the operation is defined unless the denominator is zero – division by zero is undefined. So we find the value of x where the denominator is zero:

$$x - 2 = 0$$

$$x = 2$$

So the domain is the set of all real numbers except 2.

j) Since we are taking the square root $x - 2$ cannot be less than zero. However since we are also dividing by $\sqrt{x-2}$, $(x-2)$ cannot be zero. So we have:

$$x - 2 > 0$$

$$x > 2$$

So the domain of this function is the set of real numbers such that $x > 2$.

k) Since the square root of a negative number is undefined, x has to be positive or zero. But, since division by zero is undefined the denominator cannot be zero. However, since the square root of any positive number is positive, the denominator will always be positive. Thus the domain of this function is the set of all real numbers x such that $x \geq 0$.

l) Since division by zero is undefined we need to find those values of x that would make the denominator zero.

$$(x-1)(x-2) = 0$$

$$x = 1 \ or \ x = 2$$

So the domain is the set of all real numbers except 1 and 2.

Question 8

a) We are given that the domain is the set of all positive real numbers. Now clearly the function reaches its minimum value on the given domain where x goes to zero. Also it is clear that there is no restriction on the maximum value of the function on this domain. Since $f(0) = 7$, the range of this function is $f(x) > 7$.

(Note: zero is excluded in the domain, so the range of $f(x)$ is a strict inequality.)

c) This is the same as question (a): the function reaches its minimum value on the given domain as x goes to zero and there is no restriction on its maximum value on this domain. So the range of this function is $f(x) > -1$.

d) Since x^2 is always positive, the minimum of this function is -1 where $x = 0$. There is no restriction on the maximum value on this domain. So the range is $f(x) > -1$.

f) The term $(x-1)^2$ is the square of a real number, so it is always positive. So the minimum of the function is 2 which occurs where $(x-1)^2 = 0$ (i.e. where $x = 1$). So the range is $f(x) \geq 2$.

(Note: since $x = 1$ is included in the domain, $f(x) = 2$ is included in the range.)

Question 9

a) Since x^2 is always positive, the lowest value of this function is 4, which occurs when $x = 0$. As x increases in the positive direction and in the negative direction the value of the function also increases. So the range is $f(x) \geq 4$.

c) Similarly to question (a), the minimum value of $(x-1)^2$ is zero since $(x-1)^2$ is the square of a real number. (This minimum value occurs when $x = 1$.) Therefore, $f(x)$ has a minimum of 6, and the range is $f(x) \geq 6$.

d) In the same way, the term $(1-x)^2$ has a minimum of zero when $x = 1$; for other values of x, $(1-x)^2 > 0$. Therefore $-(1-x)^2$ has a maximum of zero where $x = 1$, and is negative elsewhere. Thus, $f(x)$ has a maximum of 7 and the range is $f(x) \leq 7$.

f) The term $2(x+2)^4$ has a minimum of zero where $x = -2$ and for all other values of x it is greater than zero. So the minimum of the function is -1 and the range is $f(x) \geq -1$.

Question 10

b) This function is a straight line with a negative gradient. So to find the range of the function, we evaluate the function at the end points of the given domain:

$$f(-2) = 3 - 2(-2) = 3 + 4 = 7$$

$$f(2) = 3 - 2(2) = 3 - 4 = -1$$

So the range of f on the given domain is $-1 \leq f(x) \leq 7$.

c) x^2 is always greater than or equal to zero, so this function reaches its minimum value of zero where $x = 0$ which is included in the given domain. The greatest value of the function on this domain is $f(4) = 4^2 = 16$. So the range of the function on this domain is $0 \leq f(x) \leq 16$.

(Note: in this case we cannot simply evaluate the function at the end points of the domain since $f(-1) = (-1)^2 = (1)^2 = 1$ and $f(0) = 0 < f(-1)$. So here the minimum of the function does not occur at the left end point of the domain.)

Question 11

a) Since the power of x is even, $f(x)$ is greater than or equal to zero for all real numbers x. So its minimum value is 0 where $x = 0$. So the range is $f(x) \geq 0$.

c) Here the largest possible domain is the set of all real numbers, x excluding zero. In the expression x^3, the power of x is odd, so the value x^3 ranges over all real numbers. So the function $f(x) = \frac{1}{x^3}$ can be either positive or negative but it excludes zero. So the range is $f(x) < 0 \; or \; f(x) > 0$.

e) The term x^4 is greater than or equal to zero for all values of x. So the minimum of this function is 5 and there is no restriction on its maximum. So the range is $f(x) \geq 5$.

g) The largest possible domain for this function is the set of all real numbers such that

$$4 - x^2 \geq 0$$

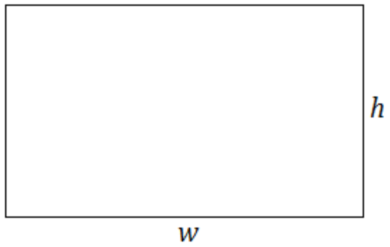
$$x^2 \leq 4$$

$$x \geq -2 \; and \; x \leq 2$$

Now $f(\pm 2) = \sqrt{4-4} = 0$ and $f(0) = \sqrt{4-0} = 2$. So the range of the function is $0 \leq f(x) \leq 2$.

Question 12

Start by drawing a diagram of the rectangle. Here we have labelled the width w and the height h.



Since the length of the wire is $24\,cm$, we have a condition on the perimeter of the rectangle:

$$2w + 2h = 24$$

$$2h = 24 - 2w$$

$$h = 12 - w \qquad (1)$$

So we can find an expression for the area:

$$A = wh = w(12 - w)$$
$$= 12w - w^2 \qquad (2)$$

We can prove that this is equal to $36 - (6-w)^2$ by rearranging the latter expression:

$$A = 36 - (6-w)^2$$

$$= 36 - (36 - 12w + w^2)$$

$$= 12w - w^2$$

Therefore:

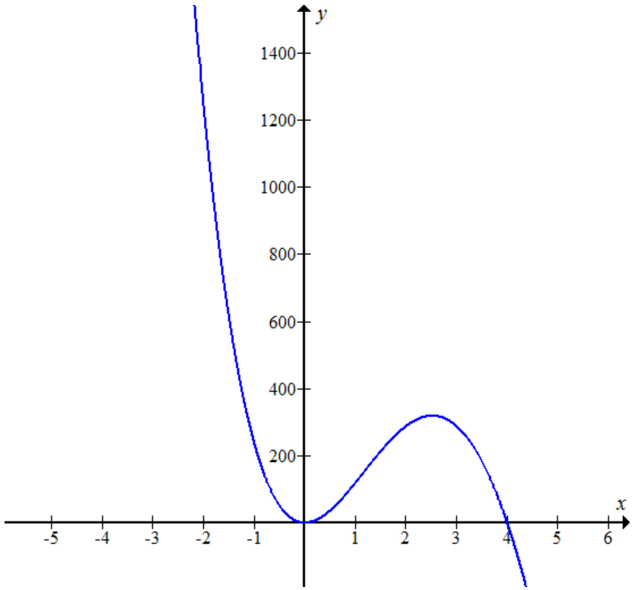
$$A = 36 - (w-6)^2 \qquad (3)$$

In order to form a rectangle, both w and h have to be greater than zero. From eqn (1) we see that this requires that $w < 12$. So the domain of the function is $0 < w < 12$. The term $(w-6)^2$ is zero where $w = 6$. So the maximum of A is 36 where $w = 6$ and the minimum is zero where $w = 0\ or\ 12$. So the range is $0 < A \leq 36$.

(Note the strict inequality on the left-hand side of the range since the domain does not include 0 or 12 and the weak inequality on the right-hand side since the domain does include 6.)

Question 13

If you expand the expression for y, you will find that the highest power of x is 3. So the graph has the shape of a cubic function.



The height, length and width of a box have to be greater than zero. This leads to three conditions on x:

$$x > 0$$

$$22 - 2x > 0 \Rightarrow x < 11$$

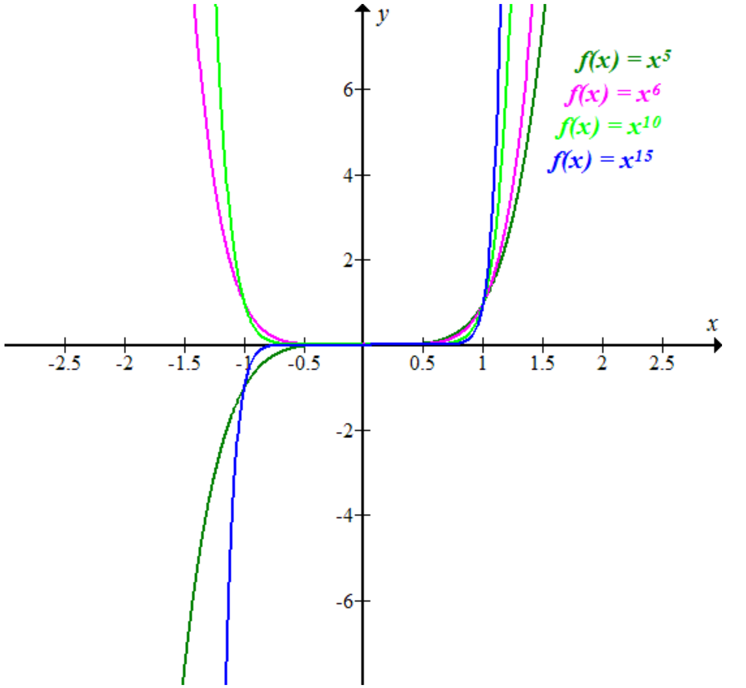
$$8 - 2x > 0 \Rightarrow x < 4$$

So an appropriate domain for this function is the set of all real numbers x such that $0 < x < 4$.

EXERCISE 3B

Question 1

The four graphs have been drawn on the same axes to show the relative characteristics of these functions.



From the graph above we can note the following characteristics of functions of the form $f(x) = x^k$ where k is a positive integer:

1. Even powers of x are always positive while odd powers of x have the same sign as x.
2. All the graphs pass through the point $(1,1)$.

3. In the region $-1 < x < 1$, for the higher the powers of x, $f(x)$ is closer to the x-axis. (I.e. the absolute value of $f(x)$ is smaller.)
4. In the regions $x < -1$ and $x > 1$, the higher the power of x the more quickly the function goes to infinity or negative infinity. (I.e. the graph is steeper.)

Question 2

To find the order in which the vertical line, $x = k$, meets the three graphs, let's substitute the value of k into the three functions:

a) $k = 2$:

$$\boldsymbol{p}\text{: } y = (2)^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

$$\boldsymbol{q}\text{: } y = (2)^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

$$\boldsymbol{r}\text{: } y = (2)^{-4} = \frac{1}{2^4} = \frac{1}{16}$$

So the points at which the vertical line $x = 2$ meets the graphs are in the following order: R, Q, P.

b) $k = \frac{1}{2}$:

$$\boldsymbol{p}\text{: } y = \left(\frac{1}{2}\right)^{-2} = 2^2 = 4$$

$$\boldsymbol{q}\text{: } y = \left(\frac{1}{2}\right)^{-3} = 2^3 = 8$$

$$\boldsymbol{r}\text{: } y = \left(\frac{1}{2}\right)^{-4} = 2^4 = 16$$

So the points at which the vertical line $x = \frac{1}{2}$ meets the graphs are in the following order: P, Q, R.

c) $k = -\frac{1}{2}$:

$$\boldsymbol{p}\text{: } y = \left(-\frac{1}{2}\right)^{-2} = (-2)^2 = 4$$

$$\boldsymbol{q}\text{: } y = \left(-\frac{1}{2}\right)^{-3} = (-2)^3 = -8$$

$$\boldsymbol{r}\text{: } y = \left(-\frac{1}{2}\right)^{-4} = (-2)^4 = 16$$

So the points at which the vertical line $x = -\frac{1}{2}$ meets the graphs are in the following order: Q, P, R.

d) $k = -2$:

$$\boldsymbol{p}\text{: } y = (-2)^{-2} = \frac{1}{(-2)^2} = \frac{1}{4}$$

$$\boldsymbol{q}\text{: } y = (-2)^{-3} = \frac{1}{(-2)^3} = -\frac{1}{8}$$

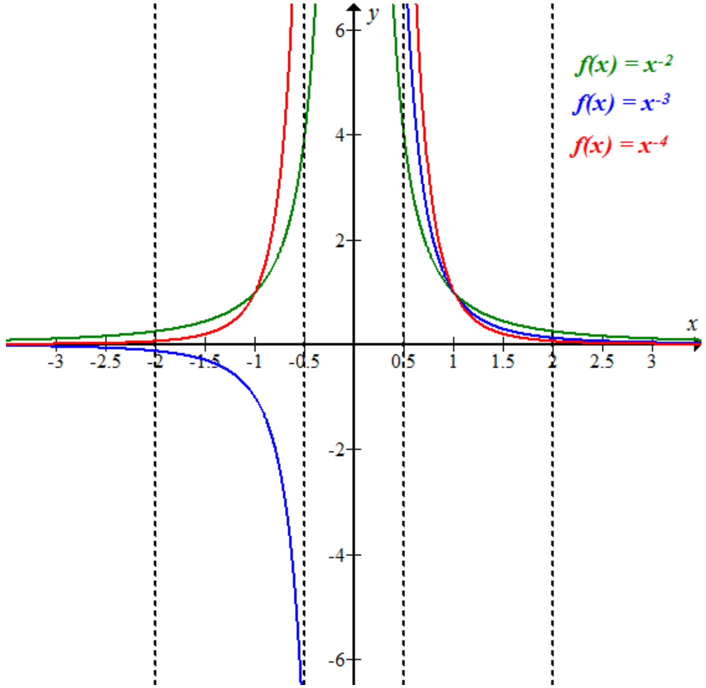
$$\boldsymbol{r}\text{: } y = (-2)^{-4} = \frac{1}{(-2)^4} = \frac{1}{16}$$

So the points at which the vertical line $x = -2$ meets the graphs are in the following order: Q, R, P.

From the characteristics of functions of the form x^k where k is a positive integer listed in question 1 and the observations above, we can work out the characteristics of functions of the form $f(x) = x^{-m}$ where m is a positive integer:

1. If m is even then $f(x)$ is symmetric about the y-axis; if m is odd then $f(x)$ is symmetric about the origin. (That is when x is negative, if m is even the graph is above the x-axis and if m is odd the graph is below the x-axis.
2. The function $f(x)$ always passes through the point $(1,1)$.
3. The function is always undefined at $x = 0$. For any m the function goes to positive infinity as x goes to zero from the right-hand side. If m is even, then the function goes to positive infinity as x goes to zero from the left-hand side and if m is odd, then the function goes to negative infinity as x goes to zero from the left.
4. As x goes to either positive or negative infinity, the function goes to zero.
5. In the region $0 < x < 1$, if m and n are positive integers and $m > n$, then $x^m < x^n$. So $\frac{1}{x^m} > \frac{1}{x^n}$. That is the graph of x^{-n} lies below the graph of x^{-m} in this region.
6. In the region $x > 1$, if m and n are positive integers and $m > n$, then $x^m > x^n$. So $\frac{1}{x^m} < \frac{1}{x^n}$. That is the graph of x^{-n} lies above the graph of x^{-m} in this region.
7. In the region $-1 < x < 0$, if m and n are positive integers and $m > n$, then $\left|\frac{1}{x^m}\right| > \left|\frac{1}{x^n}\right|$. That is the graph x^{-n} is closer to the x-axis than the graph of x^{-m}.
8. In the region $x < -1$, if m and n are positive integers and $m > n$, then $\left|\frac{1}{x^m}\right| < \left|\frac{1}{x^n}\right|$. That is the graph x^{-n} is further from the x-axis than the graph of x^{-m}.

These points are illustrated in the graph below:



Question 3

a)

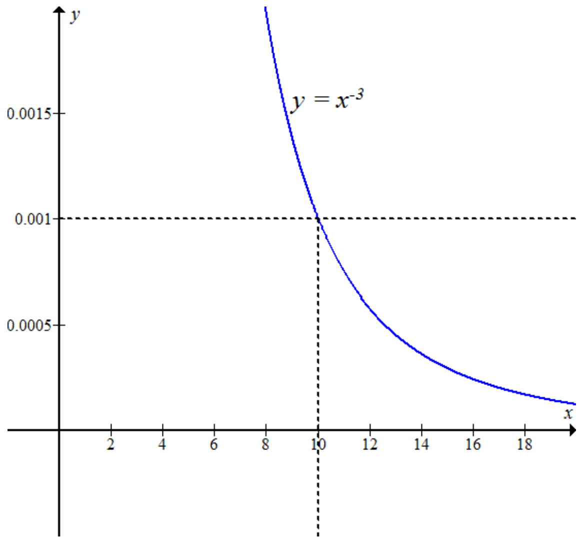
$$0 < x^{-3} \ and \ x^{-3} < 0.001$$

$$(0)^{-\frac{1}{3}} < (x^{-3})^{-\frac{1}{3}} \ and \ (x^{-3})^{-\frac{1}{3}} < (10^{-3})^{-\frac{1}{3}}$$

$$x > 0 \ and \ x > 10$$

$$\therefore x > 10$$

(Notice that the inequality sign is reversed because each side of the inequality is raised to a negative power. This is because, if $x > y$ and they both have the same sign, then $\frac{1}{x} < \frac{1}{y}$. Think about it!)



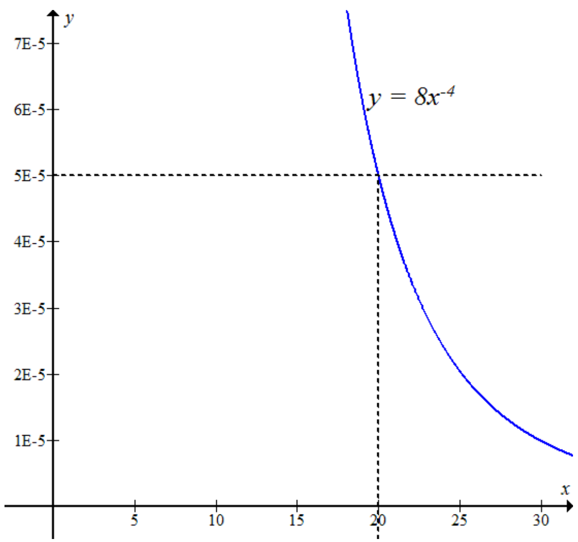
d)

$$8x^{-4} < 0.000\,05$$

$$x^{-4} < \frac{0.000\,05}{8}$$

$$(x^{-4})^{-\frac{1}{4}} > \left(\frac{0.00005}{8}\right)^{-\frac{1}{4}} = 20$$

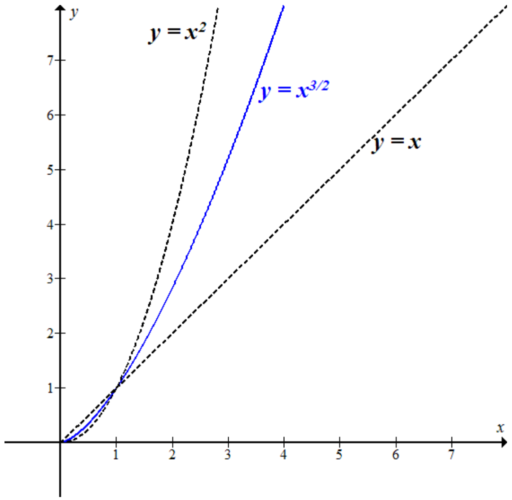
$$\therefore\ x > 20$$



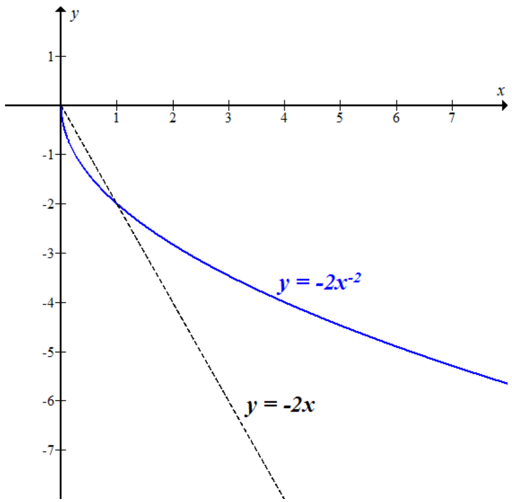
Question 4

The properties described in questions 1 and 2 apply here. Some other graphs with integer powers of x have been plotted for reference.

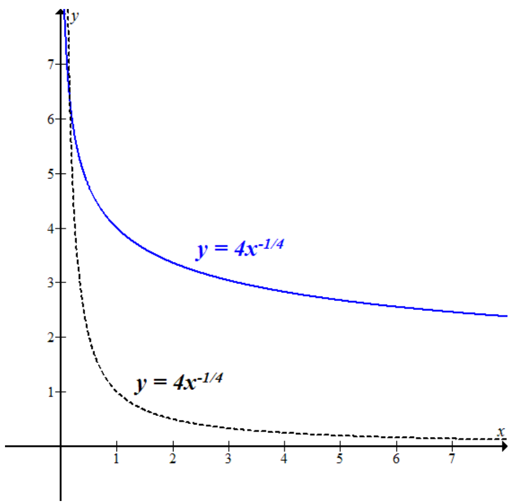
a)



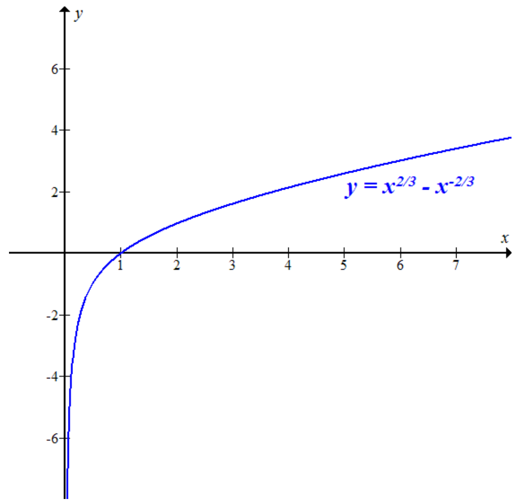
c)



d)

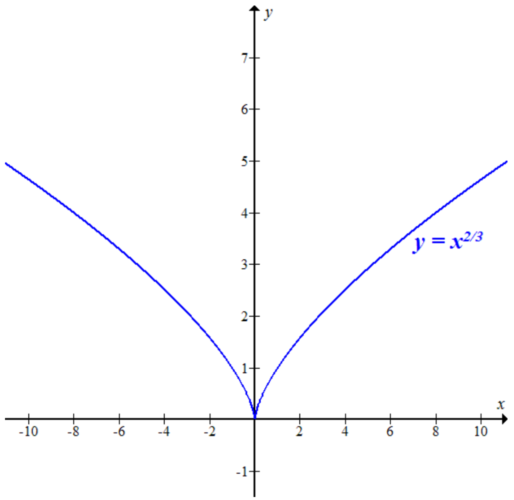


f)

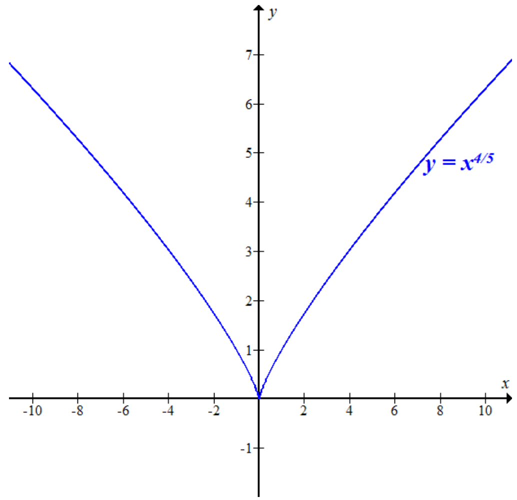


Question 5

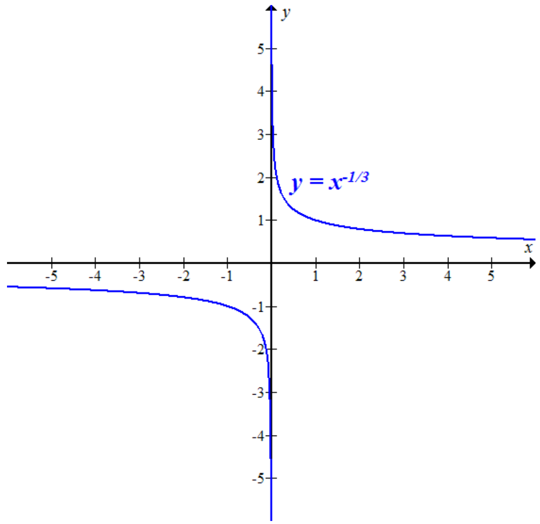
a) This function can be rewritten as $y = \sqrt[3]{x^2}$. Since x^2 is always positive, this function is defined for $x < 0$ and it is symmetric about the y-axis.



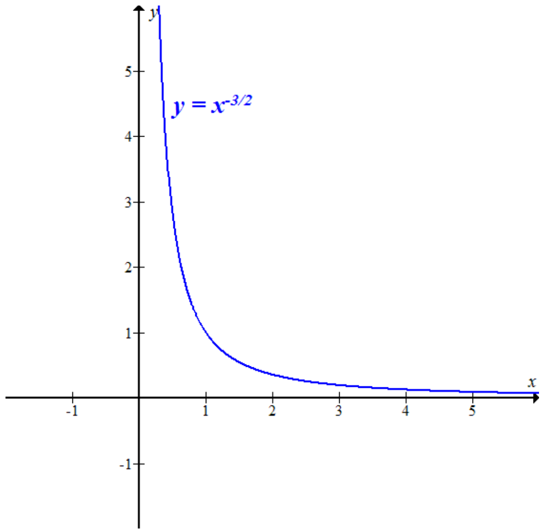
c) This function can be rewritten as $y = \sqrt[5]{x^4}$. Again, since x^4 is always positive, this function is defined for $x < 0$ and is symmetric about the y-axis. It has a similar shape to the graph in question a above, except that it goes to infinity more slowly.



d) Here the function can be written as $y = \frac{1}{\sqrt[3]{x}}$. Since it involves the cube root, it is defined for $x < 0$. However, it is not symmetric about the y-axis. It is an odd function, and symmetric about the origin.

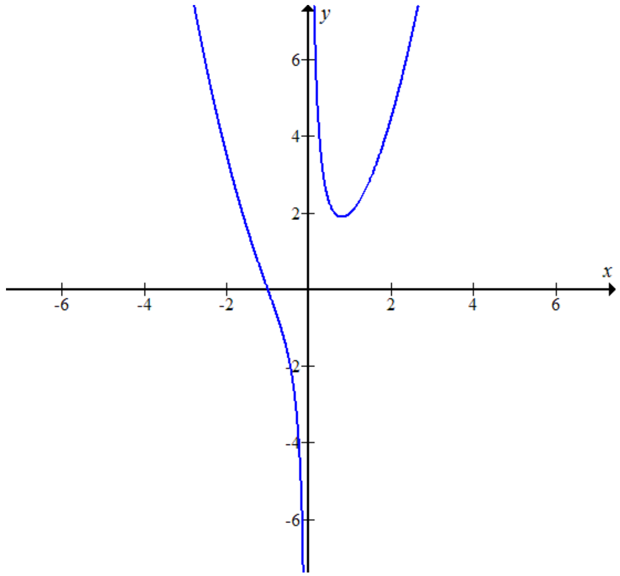


f) This function can be written as $y = \frac{1}{\sqrt{x^3}}$. This involves the square root of a number that can be either positive or negative. So it does not exist for $x < 0$.

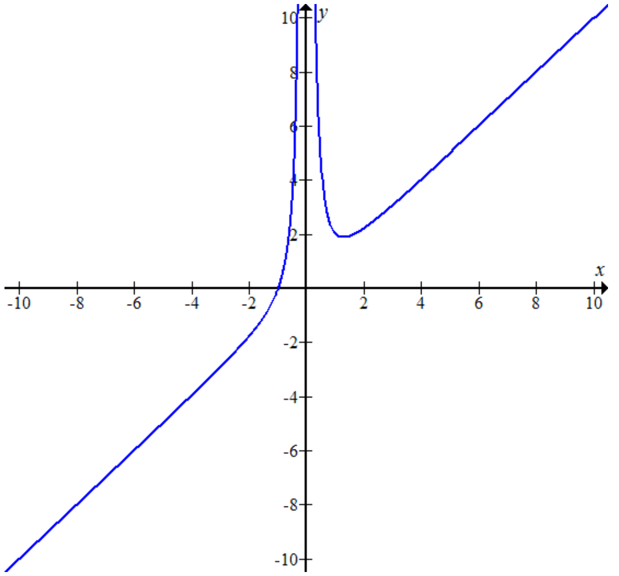


Question 6

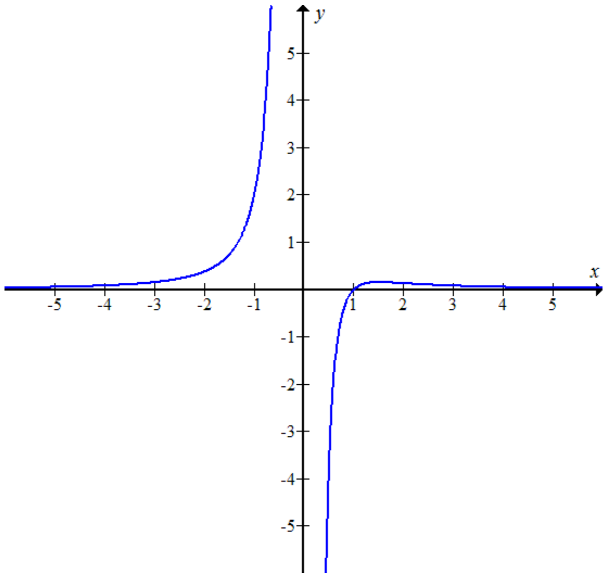
a) To sketch this function, note that at $x = 1$ both terms are equal to 1. Between $x = 0$ and $x = 1$, the term x^{-1} dominates the function and the graph goes to plus infinity as x goes to zero. To the right of $x = 1$, the term x^2 dominates the function and the graph looks like that of the function x^2. At $x = -1$, $x^2 = 1$ and $x^{-1} = -1$. So at $x = -1$ the graph crosses the y-axis. Between $x = -1$ and $x = 0$, the term x^{-1} again dominates the function and the graph to minus infinity as x goes to zero from the left. To the left of $x = -1$, the term x^2 dominates the function and it again looks like the graph of x^2.



b) At $x = 1$ both terms are equal to 1 so $y = 2$. At $x = -1$ the first term is equal to -1 while the second term is equal to 1 so $y = 0$. In the interval $-1 < x < 1$, the term x^{-2} dominates the function and the graph goes to plus infinity as x goes to zero from the left and the right. In the two regions $x < -1$ and $x > 1$, then the term x dominates the function and the graphs looks like that of a straight line.



e) In the region $-1 < x < 1$, the term x^{-3} is greater than the term x^{-2}. so in this region the graph of the function is similar to the graph of $-x^{-3}$. At $x = 1$, $y = 0$, so the graph crosses the x-axis, and at $x = -1$, $y = 2$. In the regions $x < -1$ and $x > 1$, the term x^{-2} dominates the function so the graph is similar to that of x^{-2}.



Question 7

A function $f(x)$ is an even function if for all values of x, $f(x) = f(-x)$ and a function is an odd function if for all values of x, $f(x) = -f(x)$.

a) This function is odd since x is raised to an odd power. We can also see this from the definition of an odd function:

$$y(-x) = (-x)^7 = -(x)^7 = -y(x)$$

b) This function is even since the terms involve x raised to even powers:

$$y(-x) = (-x)^4 + 3(-x)^2 = x^4 + 3x^2 = y(x)$$

c) This function is odd. It can be rewritten as $y = x^3 - x$ which shows that the two terms involve x raised to odd powers:

$$y(-x) = (-x)^3 - (-x) = -(x)^3 - (-x) = -(x^3 - x) = y(-x)$$

Question 8

a) $|-7| = 7$

b) $\left|-\frac{1}{200}\right| = \frac{1}{200} = 0.005$

c) $|9-4| = |5| = 5$

d) $|4-9| = |-5| = 5$

e) $|\pi - 3| = (\pi - 3) \quad \because \ \pi > 3$

f) $|\pi - 4| = (4 - \pi) \quad \because \ \pi < 4$

Note: parts (c) and (d) demonstrate that the modulus $(x - a)$ gives the distance between these two numbers on the real number line. That is why $|x - a| = |a - x|$. This is also the reason why if you have actual numbers, you can rewrite the expression without taking the modulus by examining which number is greater as in the parts (e) and (f).

Question 9

a) $|2 - 2^2| = |2 - 4| = |-2| = 2$

b) $\left|\frac{1}{2} - \left(\frac{1}{2}\right)^2\right| = \left|\frac{1}{2} - \frac{1}{4}\right| = \left|\frac{1}{4}\right| = \frac{1}{4}$

c) $|1 - 1^2| = |1 - 1| = 0$

d) $|-1 - (-1)^2| = |-1 - 1| = |-2| = 2$

e) $|0 - 0^2| = |0| = 0$

Question 10

a) Here we are given that $|x| > 100$, so $x^2 > (100)^2$. Therefore

$$0 < y < \frac{1}{(100)^2}$$

Since $y = x^{-2}$, x is raised to a negative power, the inequality sign is reversed. Since it is an even power of x, y will be positive (i.e. greater than zero). Simplifying the inequality gives

$$0 < y < 0.0001$$

b) Here we are given that $|x| < 0.01$. Again the inequality sign is reversed:

$$y > \frac{1}{0.01^2}$$

$$y > 10\,000$$

We do not need to state that $y > 0$ because $y > 10000$ implies that $y > 0$.

Question 11

a) Here we are given that $|x| < 1000$ and that $y = x^{-3}$. We are dealing with an odd power of x, so we have to be more careful in our working.

Note that $|x| < 1000$ can be written as

$$-1000 < x < 1000$$

(Think about this carefully: the distance of x from zero has to be less than 1000 but x can be either to the right or to the left of zero.)

Now we must consider two cases: Case I, where $x < 0$ (i.e x is negative), and Case II, where $x > 0$, i.e. x is positive.

Case I ($x > -1000$ and x is negative):

$$-1000 < x \text{ and } x < 0$$

$$-\frac{1}{1000} > \frac{1}{x} \text{ and } \frac{1}{x} < 0$$

Note that the first inequality is reversed because have inverted both sides and we are dealing with non-zero numbers (test this out with a few actual numbers). We cannot invert the second inequality because that would involve dividing by zero. But we can say that if x is negative, then $\frac{1}{x}$ is also negative, i.e. $\frac{1}{x} < 0$. Now this gives that

$$\frac{1}{x^3} < -\frac{1}{1000^3} \text{ and } \frac{1}{x^3} < 0$$

$$\therefore y < -\frac{1}{10^9} \text{ and } y < 0$$

which can be written simply as

$$y < -\frac{1}{10^9}$$

Case II ($x > 1000$ and x is positive):

$$x < 1000 \text{ and } x > 0$$

$$\frac{1}{x} > \frac{1}{1000} \text{ and } \frac{1}{x} > 0$$

$$\frac{1}{x^3} > \frac{1}{10^9} \text{ and } \frac{1}{x^3} > 0$$

$$\therefore y > \frac{1}{10^9}$$

These inequalities mean that y is further away from zero than the points -10^{-9} and 10^{-9}. In other words, the first inequality states that y is to the left of -10^{-9} and the second inequality states that y is to the right of 10^{-9}. (Draw this on a number line, if you are struggling to visualize this.) Now these two inequalities can be stated in a single statement using the modulus:

$$|y| > 10^{-9}$$

b) Now we are given that $|y| > 1000$. We follow the same method as part (a):

$|y| > 1000$ can be stated as

$$y < -1000 \text{ or } y > 1000$$

Once again, we must consider two cases, $y < -1000$ and $y > 1000$.

Case I ($y > 1000$):

$$y > 1000$$

$$\Rightarrow 0 < \frac{1}{y} < \frac{1}{1000} \qquad (1)$$

It is essential to include the proviso that $\frac{1}{y} > 0$ because we are given that $y > 1000$, meaning that y is positive. If we did not include $0 < \frac{1}{y}$ the statement $\frac{1}{y} < \frac{1}{1000}$ would allow for negative values of $\frac{1}{y}$, and these would not satisfy the original condition that $y > 1000$.

Now we have that $y = \frac{1}{x^3}$, so $x = \frac{1}{y^{\frac{1}{3}}}$. We have from (1):

$$0 < \frac{1}{y^{\frac{1}{3}}} < \frac{1}{1000^{\frac{1}{3}}}$$

$$0 < x < \frac{1}{10} \qquad (2)$$

Case II ($y < -1000$):

$$y < -1000$$

$$0 > \frac{1}{y} > -\frac{1}{1000} \qquad (3)$$

(Note again that we must add $\frac{1}{y} < 0$, i.e. negative.)

From (3) we get

$$-\frac{1}{1000^{\frac{1}{3}}} < \frac{1}{y^{\frac{1}{3}}} < 0$$

$$-\frac{1}{10} < x < 0 \qquad (4)$$

Now we can combine (2) and (4) to give

$$-\frac{1}{10} < x < \frac{1}{10}$$

$$|x| < \frac{1}{10}$$

Question 12

If N is reported as $37\,000$ to the nearest thousand, it means that N lies between $36\,500$ and $37\,500$, which can be stated as

$$36\,500 < N < 37\,500$$

This can be understood as a statement that the distance from N to $37\,000$ cannot be more than 500. Using the modulus function, this can be written as

$$|N - 37\,000| < 500$$

Question 13

The difference between the marks is less than or equal to 5 regardless of which twin receives the higher mark. This is easily stated using a modulus sign as

$$|m - n| \leq 5$$

Question 14

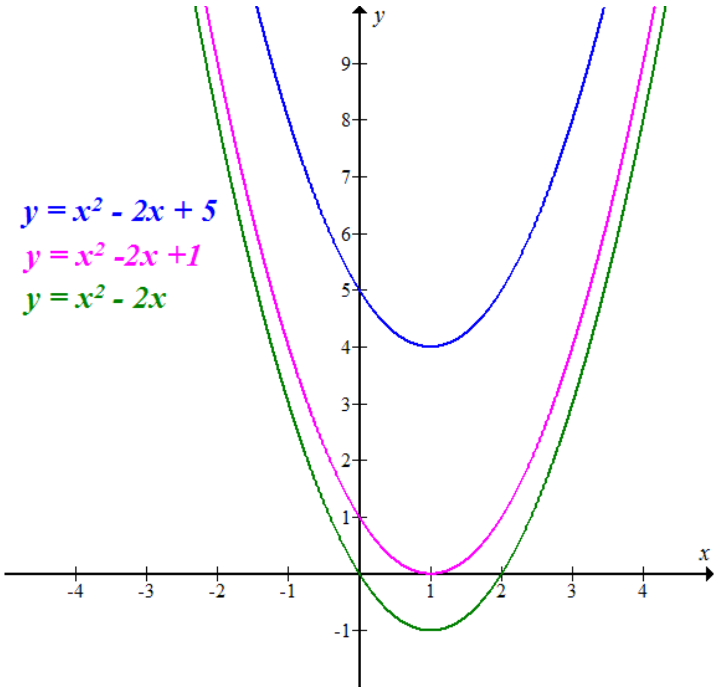
This question the same as question 12 except that it is asked the other way around.

$$|x - 5.231| < 0.005$$

means that the length x of the line is within 0.005 cm of 5.231.

EXERCISE 3C

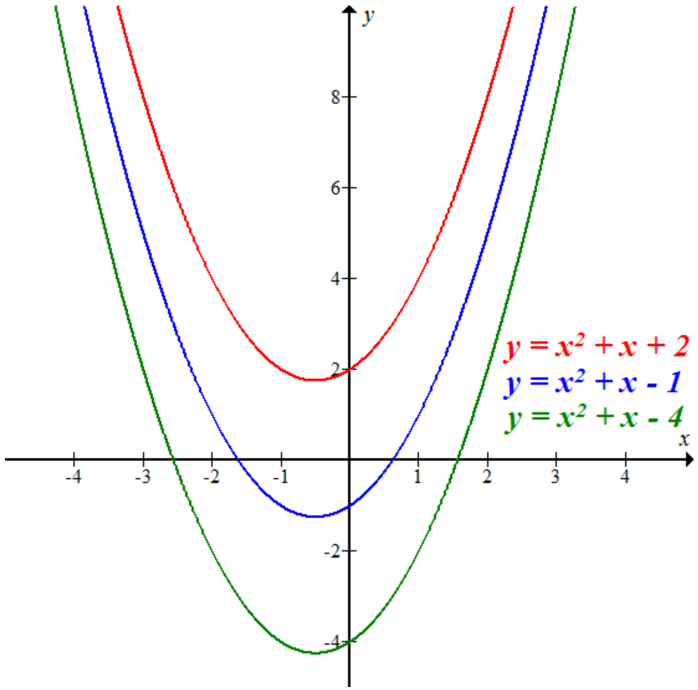
Question 1



Notice that these functions only differ in the value of c.

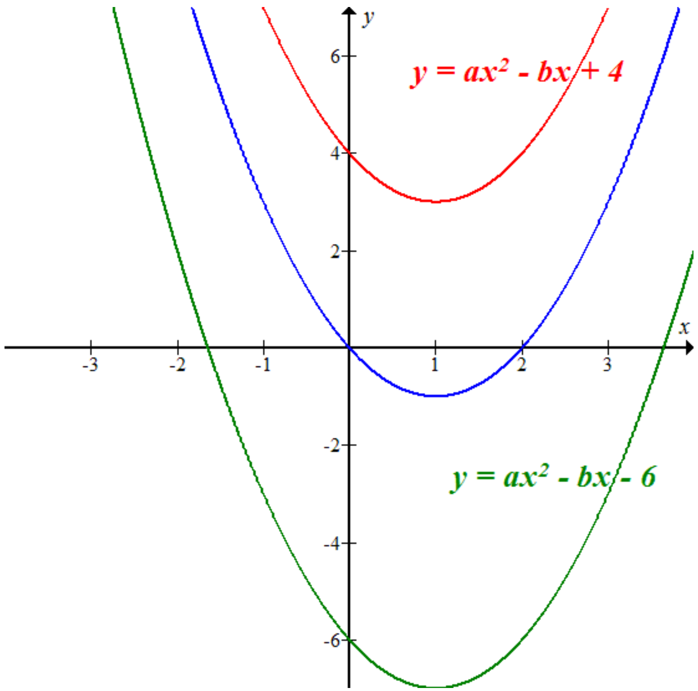
Question 2

These graphs again show how varying the value of c changes the shape of the graph.



Question 3

The given graph has $c = 0$ so it crosses the y-axis at the origin. Graph **a** has $c = 4$ so it will lie above the given graph. Graph **b** has $c = -6$ so it lies below the given graph.

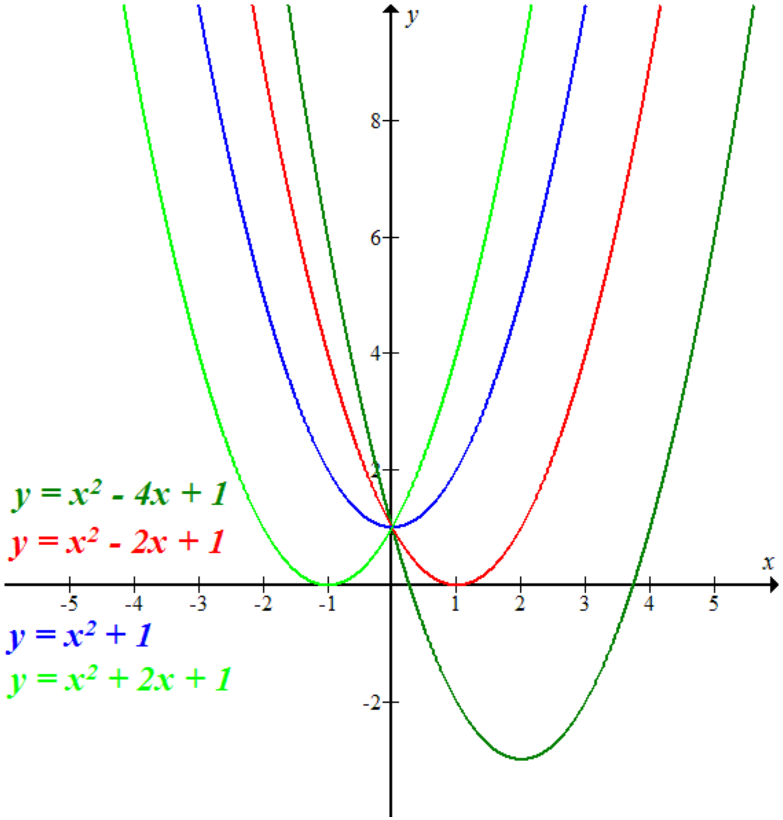


Question 4

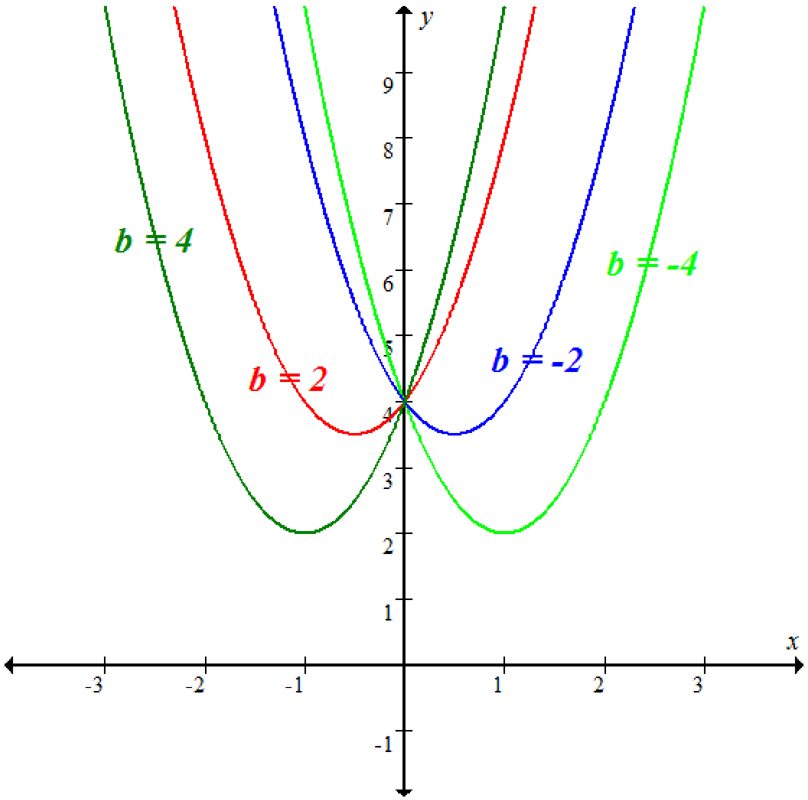
The questions above demonstrate that the effect of changing the value of c is to shift the graph along the y-axis. The value of c also determines the y-intercept of the graph.

Question 5

The following graphs differ only in the value of for b.



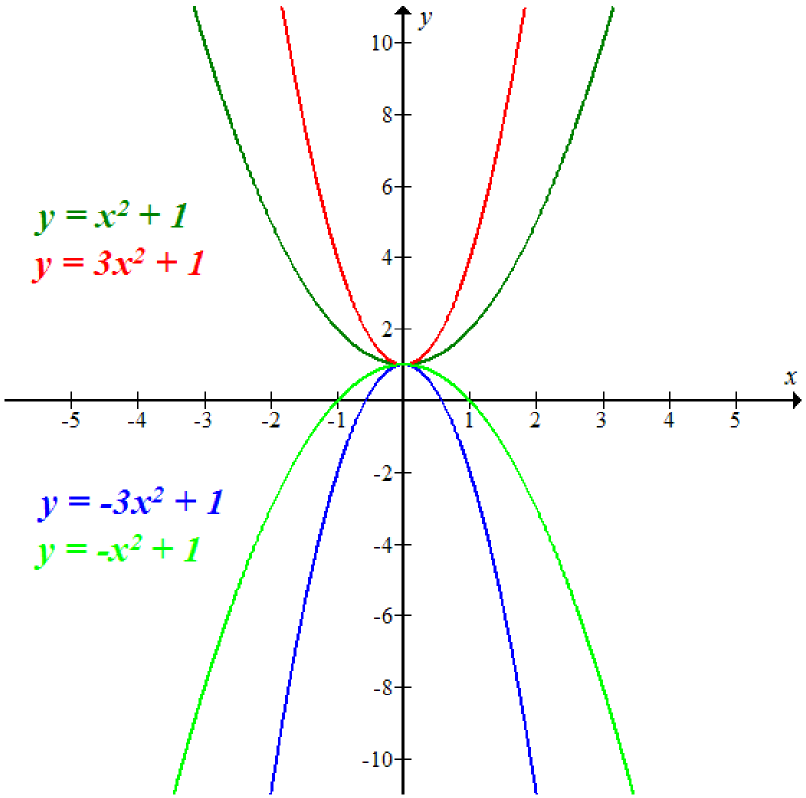
Question 6



The figure above shows the graph of $y = 2x^2 + bx + 4$ for four different values of b. This and question 5 show that the effect of changing the value of b is to shift the axis of symmetry of the graph along the x-axis. If a and b have the same sign, then the axis of symmetry is to the left of the origin and if a and b have the opposite sign axis of symmetry is to the right of the origin.

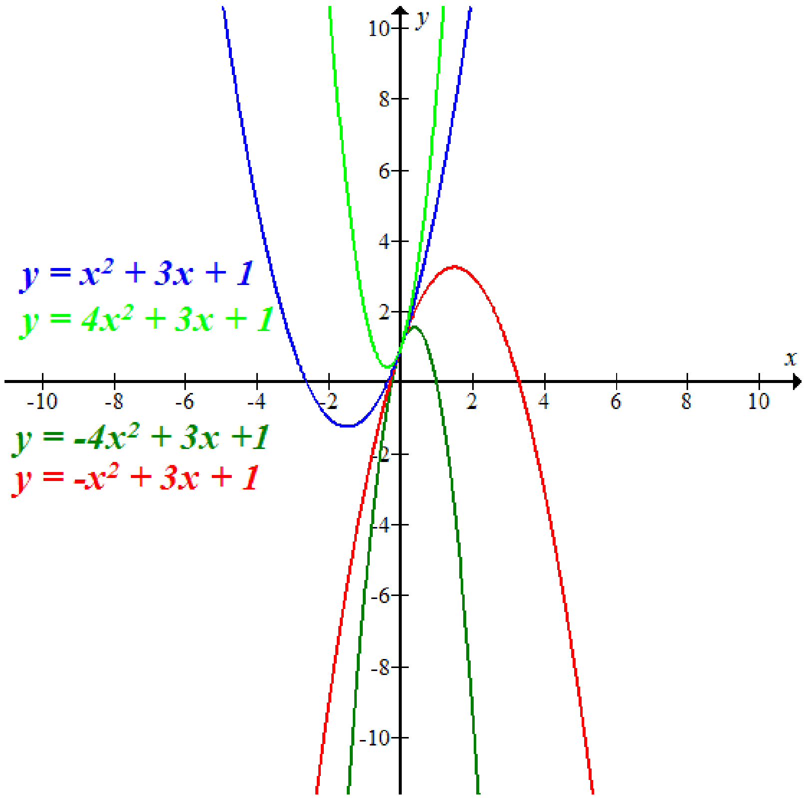
Question 7

Here the graphs all have different values for a:



Question 8

Again, these graphs only differ in the value of a:



Question 9

The value of a determines how elongated the graph is. The greater the value of $|a|$ the more the graph is elongated or lengthened in the y-direction. Furthermore if a is positive the vertex is at the minimum point of the graph and if a is negative the vertex is at the maximum point of the graph.

Question 10

In this graph the vertex is at the minimum point and the axis of symmetry is to the left of the y-axis. This requires that a is positive and that a and b have the same sign. So only the curve in (c) could be the equation of the curve shown in the diagram.

Question 11

Here the vertex is at the maximum point of the curve and the axis of symmetry is to the right of the y-axis. So this requires that a is negative and that a and b have the opposite sign. Only the curve in (a) satisfies these requirements.

EXERCISE 3D

Question 1

a)

$$x = 3 \qquad (1)$$

$$y = x^2 + 4x - 7 \qquad (2)$$

(1) → (2):

$$y = (3)^2 + 4(3) - 7$$

$$y = 9 + 12 - 7$$

$$y = 14 \qquad (3)$$

So the curve $y = x^2 + 4x - 7$ intersects the vertical line $x = 3$ at the point $(3, 14)$.

d)

$$y + 3 = 0 \qquad (1)$$

$$y = 2x^2 + 5x - 6 \qquad (2)$$

Rearrange eqn (1):

$$y = -3 \qquad (3)$$

(3) → (2):

$$-3 = 2x^2 + 5x - 6$$

$$2x^2 + 5x - 3 = 0$$

$$(2x - 1)(x + 3) = 0$$

$$x = \frac{1}{2} \ \ or \ \ x = -3 \qquad (4)$$

The horizontal line $y = -3$ intersects the curve $y = 2x^2 + 5x - 6$ at the points $\left(\frac{1}{2}, -3\right)$ and $(-3, -3)$.

Question 2

a)

$$y = x + 1 \qquad (1)$$

$$y = x^2 - 3x + 4 \qquad (2)$$

(1) → (2):

$$x + 1 = x^2 - 3x + 4$$

$$x^2 - 4x + 3 = 0$$

$$(x - 3)(x - 1) = 0$$

$$x = 3 \ \ or \ \ x = 1 \qquad (3)$$

At $x = 3$,

$$y = 3 + 1 = 4$$

At $x = 1$,

$$y = 1 + 1 = 2$$

So the line $y = x + 1$ intersects the curve $y = x^2 - 3x + 4$ at the points $(3, 4)$ and $(1, 2)$.

c)

$$y = 3x + 11 \qquad (1)$$

$$y = 2x^2 + 2x + 5 \qquad (2)$$

(1) → (2):

$$3x + 11 = 2x^2 + 2x + 5$$

$$2x^2 - x - 6 = 0$$

$$(2x + 3)(x - 2) = 0$$

$$x = -\frac{3}{2} \;\; or \;\; x = 2$$

At $x = -\frac{3}{2}$,

$$y = 3\left(-\frac{3}{2}\right) + 11 = -\frac{9}{2} + 11 = \frac{13}{2}$$

At $x = 2$

$$y = 3(2) + 11 = 6 + 11 = 17$$

So the curve $y = 2x^2 + 2x + 5$ intersects the line $y = 3x + 11$ at the points $\left(-\frac{3}{2}, \frac{13}{2}\right)$ and $(2, 17)$.

e)

$$3x + y - 1 = 0 \qquad (1)$$

$$y = 6 + 10x - 6x^2 \qquad (2)$$

Rearrange eqn (1):

$$y = -3x + 1 \qquad (3)$$

(3) → (2):

$$-3x + 1 = 6 + 10x - 6x^2$$

$$6x^2 - 13x - 5 = 0$$

$$(3x + 1)(2x - 5) = 0$$

$$x = -\frac{1}{3} \;\; or \;\; x = \frac{5}{2}$$

At $x = -\frac{1}{3}$,

$$y = -3\left(-\frac{1}{3}\right) + 1 = 1 + 1 = 2$$

At $x = \frac{5}{2}$,

$$y = -3\left(\frac{5}{2}\right) + 1 = -\frac{15}{2} + 1 = -\frac{13}{2}$$

So the line $y = -3x + 1$ intersects the curve $y = 6 + 10x - 6x^2$ at the points $\left(-\frac{1}{3}, 2\right)$ and $\left(\frac{5}{2}, -\frac{13}{2}\right)$.

Question 3

a)

$$y = 2x + 2 \qquad (1)$$

$$y = x^2 - 2x + 6 \qquad (2)$$

(1) → (2):

$$2x + 2 = x^2 - 2x + 6$$

$$x^2 - 4x + 4 = 0$$

$$(x - 2)(x - 2) = 0$$

$$x = 2$$

At $x = 2$,

$$y = 2(2) + 2 = 6$$

So there is only one point that simultaneously satisfies equations 1 and 2. So the line $y = 2x + 2$ and the curve $y = x^2 - 2x + 6$ meet only at the point $(2, 6)$.

b)

$$y = -2x - 7 \qquad (1)$$
$$y = x^2 + 4x + 2 \qquad (2)$$

(1) → (2):

$$-2x - 7 = x^2 + 4x + 2$$

$$x^2 + 6x + 9 = 0$$

$$(x + 3)(x + 3) = 0$$

$$x = -3$$

At $x = -3$,

$$y = -2(-3) - 7 = 6 - 7 = -1$$

Again, there is only one point that satisfies both equations simultaneously. Thus, the line and the curve meet at only one point: $(-3, 1)$.

Question 4

a)

$$y = x \qquad (1)$$
$$y = x^2 - x \qquad (2)$$

(1) → (2):

$$x = x^2 - x$$

$$x^2 - 2x = 0$$

$$x(x - 2) = 0$$

$$x = 0 \ or \ x = 2$$

At $x = 0$:

$$y = 0$$

At $x = 2$:

$$y = 2$$

The two points of intersection are $(0, 0)$ and $(2, 2)$.

b)

$$y = x - 1 \qquad (1)$$
$$y = x^2 - x \qquad (2)$$

(1) → (2):

$$x - 1 = x^2 - x$$

$$x^2 - 2x + 1 = 0$$

$$(x - 1)(x - 1) = 0$$

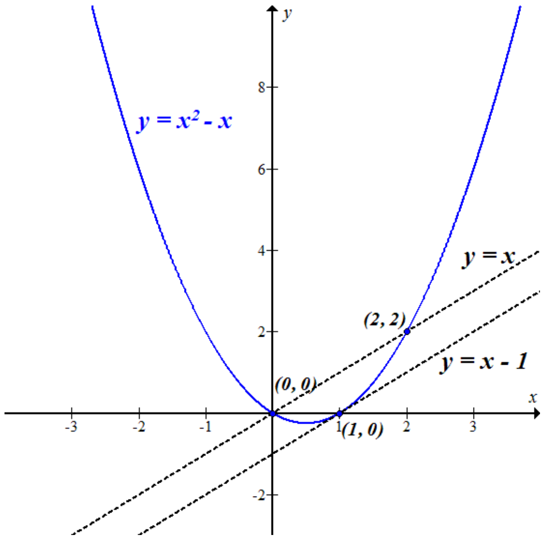
$$x = 1$$

At $x = 1$:

$$y = 1 - 1 = 0$$

So the only point of intersection is $(1, 0)$.

The curve $y = x^2 - x$ along with the two lines, $y = x$ and $y = x - 1$, are plotted below:



Question 5

a)

$$y = -3x + 2 \qquad (1)$$

$$y = x^2 + 5x + 18 \qquad (2)$$

(1) → (2):

$$-3x + 2 = x^2 + 5x + 18$$

$$x^2 + 8x + 16 = 0$$

$$(x + 4)(x + 4) = 0$$

$$x = -4$$

At $x = -4$:

$$y = -3(-4) + 2$$

$$y = 12 + 2 = 14$$

So the line and curve intersect at $(-4, 14)$.

b)

$$y = -3x + 6 \qquad (1)$$

$$y = x^2 + 5x + 18 \qquad (2)$$

(1) → (2):

$$-3x + 6 = x^2 + 5x + 18$$

$$x^2 + 8x + 12 = 0$$

$$(x + 2)(x + 6) = 0$$

$$x = -2 \;\; or \;\; x = -6$$

At $x = -2$:

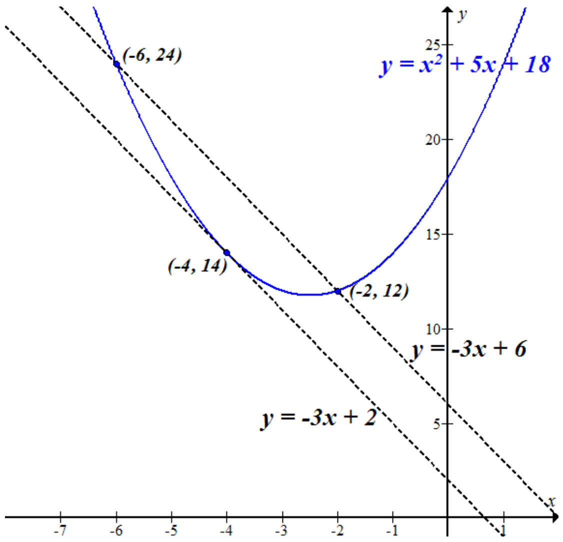
$$y = -3(-2) + 6 = 12$$

At $x = -6$:

$$y = -3(-6) + 6 = 24$$

So the line and the curve meet at the points $(-2, 12)$ and $(-6, 24)$.

The curve and the two lines are plotted below:



Question 6

a)

$$y = x + 5 \qquad (1)$$

$$y = 2x^2 - 3x - 1 \qquad (2)$$

(1) → (2):

$$x + 5 = 2x^2 - 3x - 1$$

$$2x^2 - 4x - 6 = 0$$

$$(2x - 6)(x + 1) = 0$$

$$x = 3 \ \ or\ x = -1$$

At $x = 3$:

$$y = 3 + 5 = 8$$

At $x = -1$:

$$y = -1 + 5 = 4$$

So the curve and the line intersect at the points $(3, 8)$ and $(-1, 4)$.

b)

$$y = x + 5 \qquad (1)$$

$$y = 2x^2 - 3x + 7 \qquad (2)$$

(1) → (2):

$$x + 5 = 2x^2 - 3x + 7$$

$$2x^2 - 4x + 2 = 0$$

$$2(x^2 - 2x + 1) = 0$$

$$2(x - 1)(x - 1) = 0$$

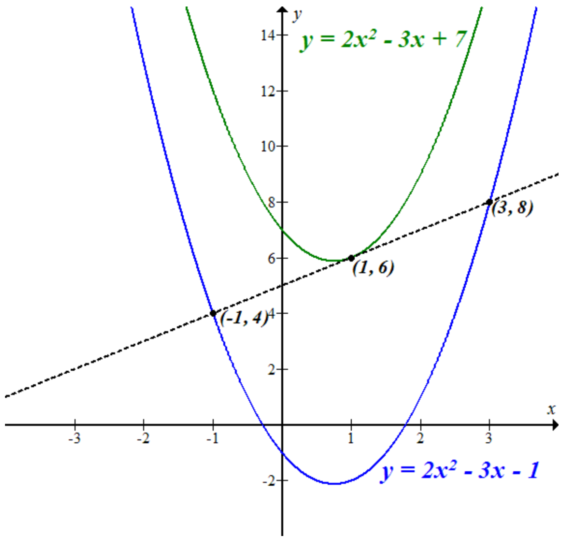
$$x = 1$$

At $x = 1$:

$$y = 1 + 5 = 6$$

The curve and the line at the point $(1, 6)$.

The curve and the two lines are plotted below:



Question 7

a)

$$y = x^2 + 5x + 1 \qquad (1)$$
$$y = x^2 + 3x + 11 \qquad (2)$$

(1) → (2):

$$x^2 + 5x + 1 = x^2 + 3x + 11$$

$$2x - 10 = 0$$

$$x = 5$$

At $x = 5$

$$y = (5)^2 + 3(5) + 11$$

$$y = 25 + 15 + 11$$

$$y = 51$$

So the two curves intersect at the point $(5, 51)$.

c)

$$y = 7x^2 + 4x + 1 \qquad (1)$$
$$y = 7x^2 - 4x + 1 \qquad (2)$$

(1) → (2):

$$7x^2 + 4x + 1 = 7x^2 - 4x + 1$$

$$8x = 0$$

$$x = 0$$

At $x = 0$:

$$y = 7(0)^2 + 4(x) + 1 = 1$$

So these two curves intersect at $(0, 1)$.

Question 8

a)

$$y = \frac{1}{2}x^2 \qquad (1)$$

$$y = 1 - \frac{1}{2}x^2 \qquad (2)$$

(1) → (2):

$$\frac{1}{2}x^2 = 1 - \frac{1}{2}x^2$$

$$x^2 = 1$$

$$x = 1 \ or \ x = -1$$

At $x = 1$:

$$y = \frac{1}{2}(1)^2 = \frac{1}{2}$$

At $x = -1$

$$y = \frac{1}{2}(-1)^2 = \frac{1}{2}$$

So the curves intersect at $\left(1, \frac{1}{2}\right)$ and $\left(-1, \frac{1}{2}\right)$.

c)

$$y = x^2 + 7x + 13 \qquad (1)$$

$$y = 1 - 3x - x^2 \qquad (2)$$

(1) → (2)

$$x^2 + 7x + 13 = 1 - 3x - x^2$$

$$2x^2 + 10x + 12 = 0$$

$$(2x + 6)(x + 2) = 0$$

$$x = -3 \ or \ x = -2$$

At $x = -3$

$$y = 1 - 3(-3) - (-3)^2 = 1 + 9 - 9 = 1$$

At $x = -2$

$$y = 1 - 3(-2) - (-2)^2 = 1 + 6 - 4 = 3$$

So the curves intersect at $(-3, 1)$ and $(-2, 3)$.

e)

$$y = (x - 2)(6x + 5) \qquad (1)$$

$$y = (x - 5)^2 + 1 \qquad (2)$$

Multiply out the factors in each expression so that it is in the form $ax^2 + bx + c$:

$$y = 6x^2 - 12x + 5x - 10$$

$$y = 6x^2 - 7x - 10 \qquad (3)$$

$$y = x^2 - 10x + 25 + 1$$

$$y = x^2 - 10x + 26 \qquad (4)$$

(4) → (3):

$$x^2 - 10x + 26 = 6x^2 - 7x - 10$$

$$5x^2 + 3x - 36 = 0$$

$$(5x - 12x)(x + 3) = 0$$

$$x = \frac{12}{5} \ or \ x = -3$$

At $x = \frac{12}{5}$:

$$y = \left(\frac{12}{5}\right)^2 - 10\left(\frac{12}{5}\right) + 26$$

$$y = \frac{144}{25} - 24 + 26$$

$$y = \frac{144}{25} + 2 = \frac{194}{25}$$

At $x = -3$,

$$y = (-3)^2 - 10(-3) + 26$$

$$y = 9 + 30 + 26$$

$$y = 65$$

So the curves intersect at the points $\left(\frac{12}{5}, \frac{194}{25}\right)$ and $(-3, 65)$.

Question 9

a)

$$y = 8x^2 \qquad (1)$$

$$y = 8x^{-1} \qquad (2)$$

(2) → (1):

$$\frac{8}{x} = 8x^2$$

$$8x^3 = 8$$

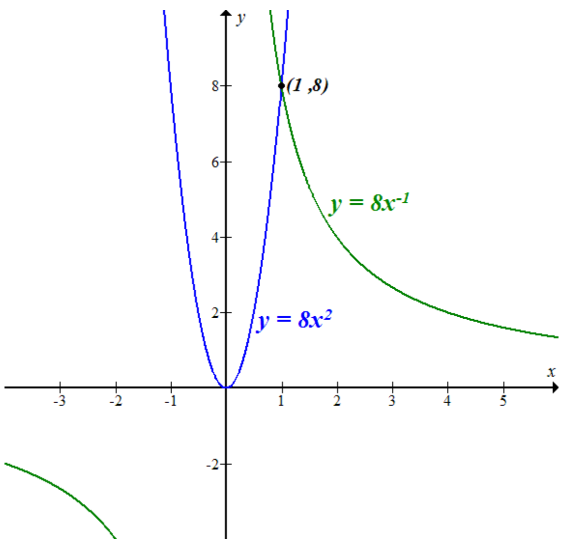
$$x^3 = 1$$

$$x = 1 \qquad (3)$$

(3) → (1):

$$y = 8(1)^2 = 8$$

So the curves intersect at the point $(1, 8)$ as illustrated in the graph below:



c)

$$y = x \qquad (1)$$

$$y = 4x^{-3} \qquad (2)$$

(1) → (2):

$$x = 4x^{-3}$$

$$xx^3 = 4$$

$$x^4 = 4$$

$$x = \sqrt{2} \ \ or \ \ x = -\sqrt{2} \qquad (3)$$

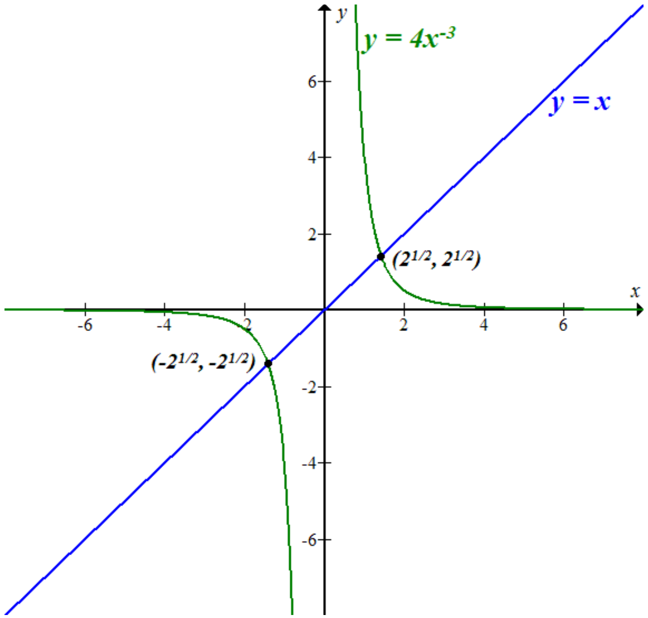
At $x = \sqrt{2}$

$$y = \sqrt{2}$$

At $x = -\sqrt{2}$

$$y = -\sqrt{2}$$

So the line $y = x$ intersects the curve $y = 4x^{-3}$ at the points $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$.



e)

$$y = 9x^{-3} \quad (1)$$

$$y = x^{-5} \quad (2)$$

(2) → (1):

$$x^{-5} = 9x^{-3}$$

$$x^3x^{-5} = 9$$

$$x^{-2} = 9$$

$$x = 9^{-\frac{1}{2}}$$

$$x = \pm\frac{1}{3}$$

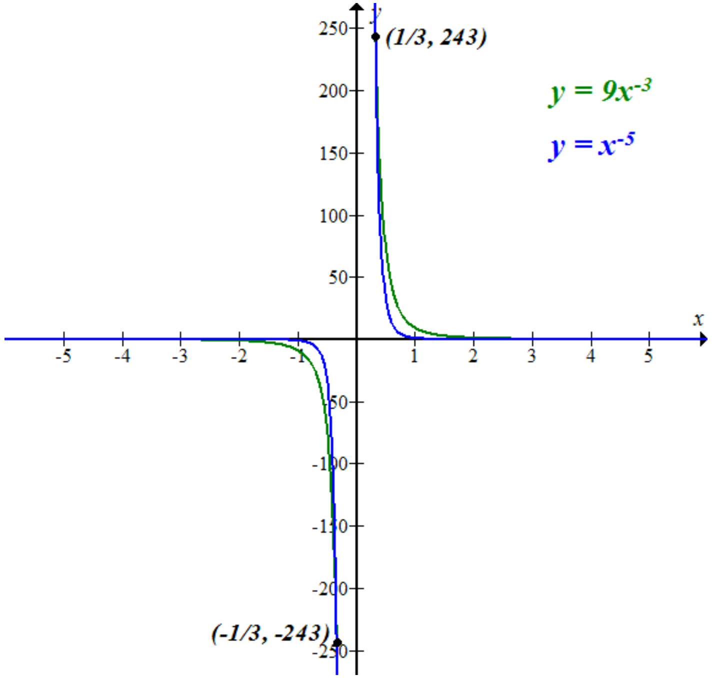
At $x = \frac{1}{3}$

$$y = \left(\frac{1}{3}\right)^{-5} = 3^5 = 243$$

At $x = -\frac{1}{3}$

$$y = \left(-\frac{1}{3}\right)^{-5} = (-3)^5 = -243$$

So the curves intersect at the points $\left(\frac{1}{3}, 243\right)$ and $\left(-\frac{1}{3}, -243\right)$, as illustrated in the graph below:



EXERCISE 3E

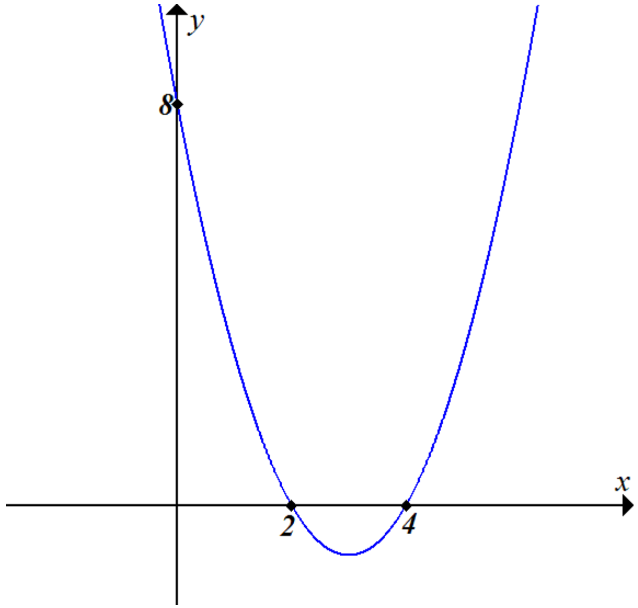
Question 1

a) First, we need to find the values of x when $y = 0$:

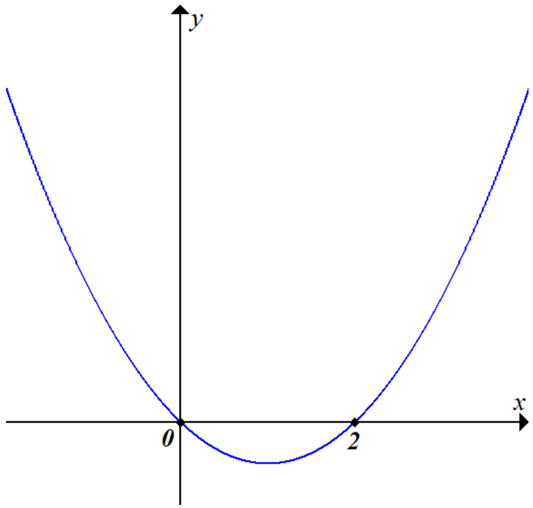
$$(x-2)(x-4) = 0$$

$$x = 2 \text{ or } x = 4$$

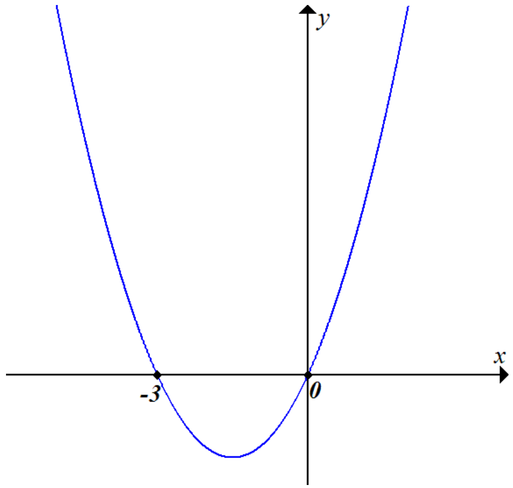
So the graph passes through the points $(2, 0)$ and $(4, 0)$. By inspection it is easy to see that in the expanded expression the coefficient x^2 will be $+1$. So the graph faces upwards and is not elongated. Furthermore, the graph intersects the y-axis at $(0, 8)$.



c) This graph passes through the points $(0,0)$ and $(2,0)$. It also faces upwards and crosses the y- axis at 0:

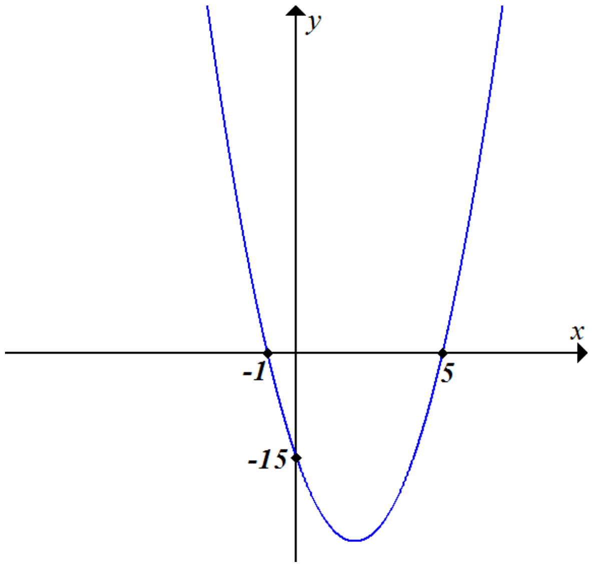


e) This graph passes through the points $(0,0)$ and $(-3,0)$ and also faces upwards:

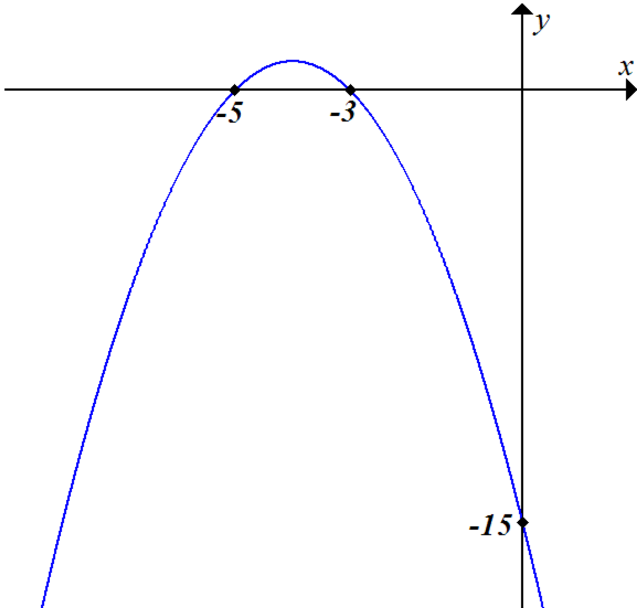


Question 2

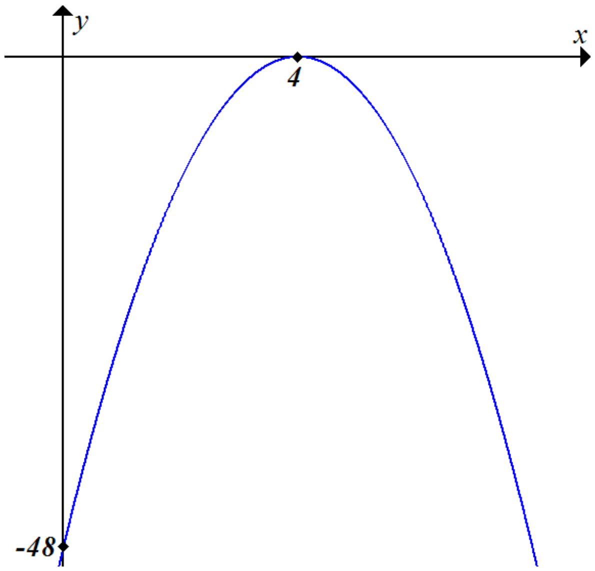
a) This graph passes through the points $(-1,0)$ and $(5,0)$ and the y-intercept is $(0,-15)$. It faces upwards and is elongated since the coefficient of x^2 is 3:



c) This graph passes through the points $(-3,0)$ and $(-5,0)$ and it crosses the y-axis at $(0,-15)$. Since the coefficient of x^2 is -1, it faces downwards but is not elongated:



e) This graph only touches the x-axis at one point $(4,0)$ which has to be the vertex of the parabola. Because of the coefficient of x^2 is -3, the graph faces downwards and is elongated. It crosses the y-axis at $(0,-48)$:

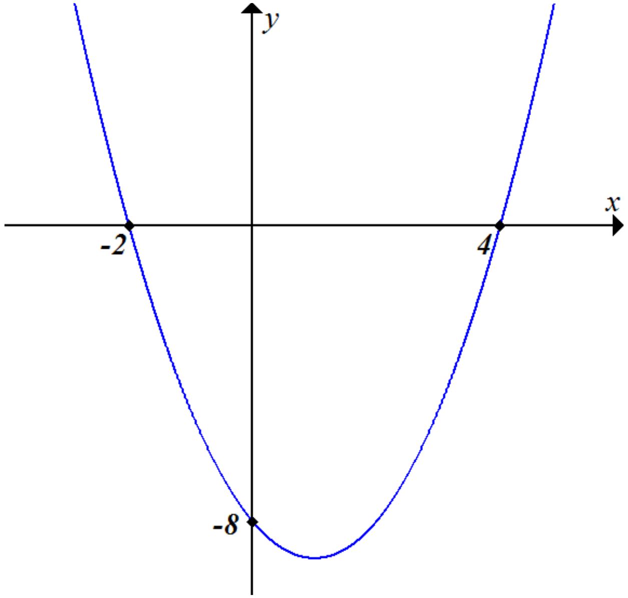


Question 3

a) We first factorise the function:

$$y = (x-4)(x+2)$$

So the graph passes through $(-2,0)$ and $(4,0)$ and the y-intercept is at $(0,-8)$. Since the coefficient of x^2 is 1 the graph faces upwards and is not elongated:



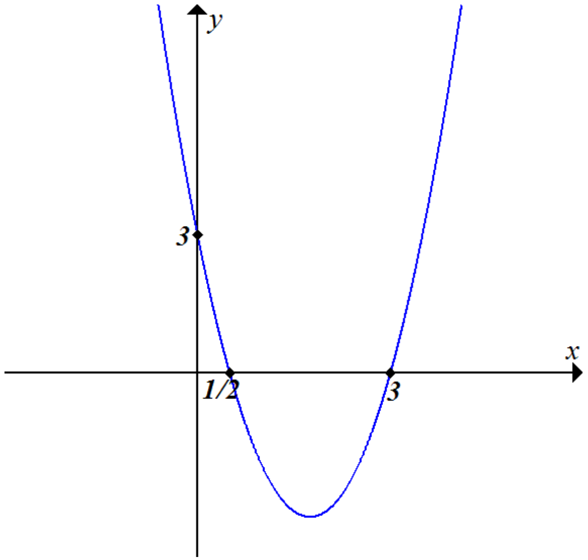
d)

$$y = 2x^2 - 7x + 3$$

$$y = 2\left(x^2 - \frac{7}{2}x + \frac{3}{2}\right)$$

$$y = 2(x-3)\left(x - \frac{1}{2}\right)$$

So the graph passes through the points $\left(\frac{1}{2},0\right)$ and $(3,0)$ and crosses the y-axis at $(0,3)$. It faces upwards an is somewhat elongated:



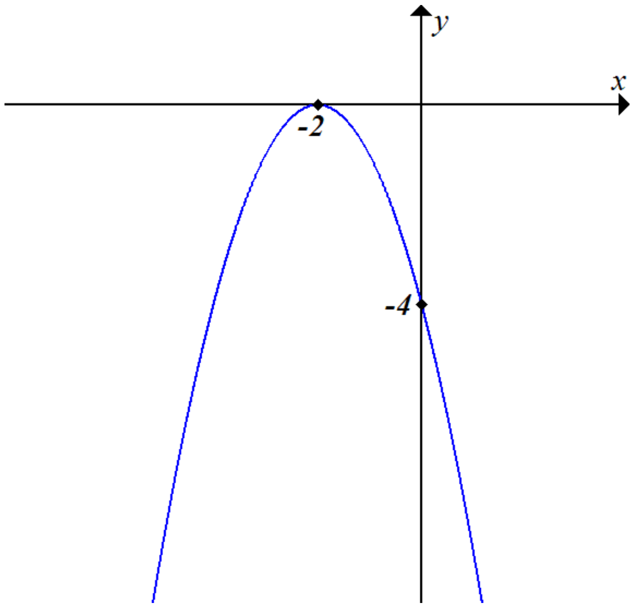
g)

$$y = -x^2 - 4x - 4$$

$$y = -(x^2 + 4x + 4)$$

$$y = -(x+2)(x+2)$$

Here the graph touches the x-axis at the point $(-2,0)$ and crosses the y-axis at $(0,-4)$. It faces downwards and is not elongated:



Question 4

a) Since the we know the values of x at which y is zero we can work out the factored form of the equation:

$$y = (x-2)(x-5)$$

Now we can multiply it out to get the equation in the form $y = ax^2 + bx + c$.

$$y = x^2 - 5x - 2x + 10$$

$$y = x^2 - 7x + 10$$

c)

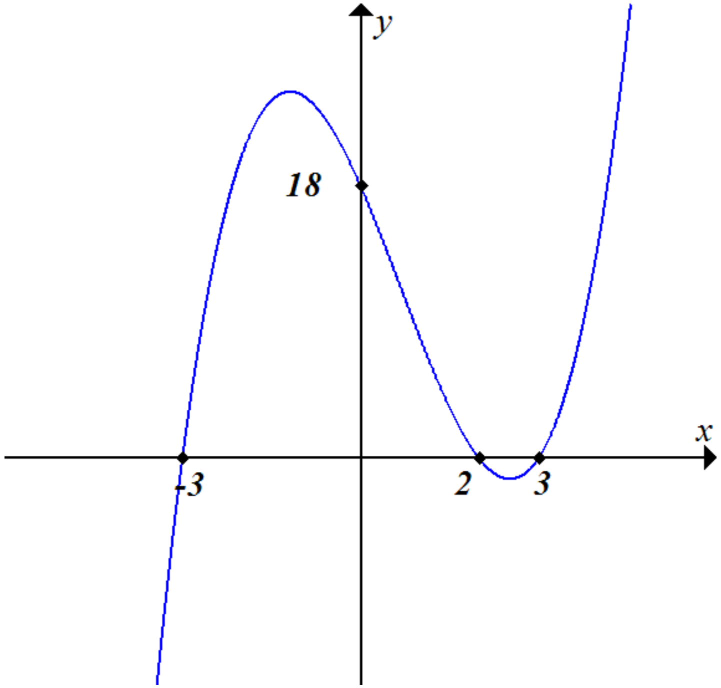
$$y = (x+5)(x-3)$$

$$y = x^2 - 3x + 5x - 15$$

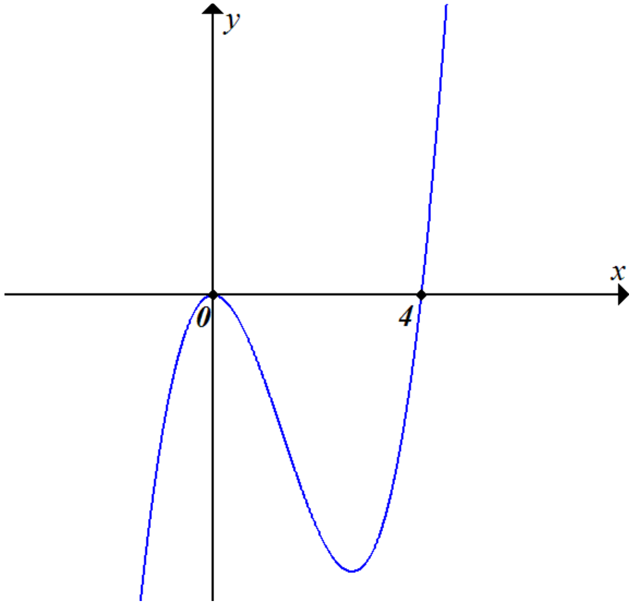
$$y = x^2 + 2x - 15$$

Question 5

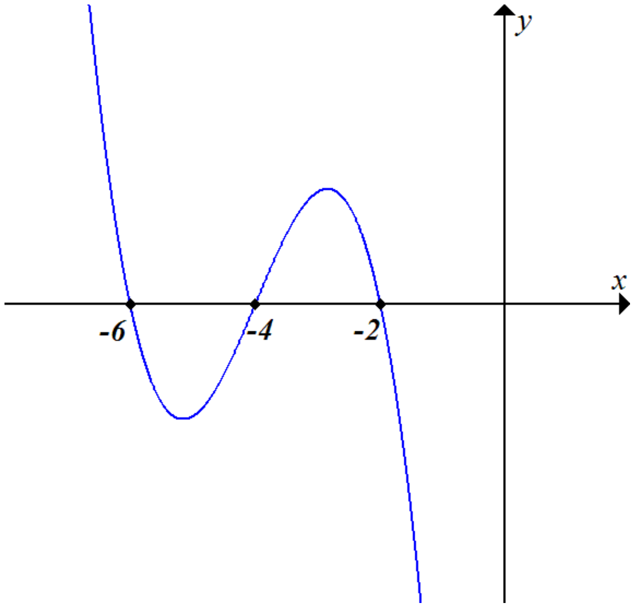
a) This function has three factors so it is going to be a cubic. It crosses the x-axis at the three points $(-3,0)$, $(2,0)$, and $(3,0)$. The constant 1 in front of the factors is implied and since it is positive the graph goes to positive infinity as x becomes very large (i.e. the graph lies in the first quadrant as x becomes very large). Since the graph goes to positive infinity for very large x, it comes from negative infinity for very small x:



c) Here the first factor x is squared, so at the point $x = 0$, the graph touches the x-axis so that the x-axis is tangent to the graph at that point. From the other factor $(x - 4)$ we know that the graph crosses the x-axis at the point $(4,0)$. Again the coefficient before the factors is $+1$ so the graph goes to positive infinity as x goes to infinity and the graph goes to negative infinity as x goes to negative infinity:



e) This graph crosses the x-axis at the points $(-6,0)$, $(-4,0)$ and $(-2,0)$. It crosses the y-axis at -48. Because the coefficient before the factors is -1, the graph is opposite to those above: as x goes to negative infinity, the graph goes to positive infinity and as x goes to infinity, the graph goes to negative infinity.



Question 6

a) Here we can again find the factored from of the equation of a parabola that passes through the given points.

$$y = (x - 1)(x - 5)$$

$$y = x^2 - 6x + 5$$

But we were given that the parabola crosses the y-axis at the point $(0,15)$, so we have to multiply our equation by 3 to obtain the correct result:

$$y = 3x^2 - 18x + 15$$

c) The equation of a parabola that passes through the given points is

$$y = (x+6)(x+2)$$

$$y = x^2 + 8x + 12$$

Now we are given that the graph also passes through the point $(0,-6)$, so we need to multiply the equation we found by $-\frac{1}{2}$:

$$y = -\frac{1}{2}x^2 - 4x - 6$$

e) The equation of a parabola that passes through $(-10,0)$ and $(7,0)$ is

$$y = (x+10)(x-7)$$

$$y = x^2 + 3x - 70$$

But the graph must also pass through the point $(8,90)$, so we need to multiply the equation by a factor a such that the point $(8,90)$ lies on the curve:

$$y = a(x^2 + 3x - 70)$$

$$90 = a(8^2 + 3(8) - 70)$$

$$90 = a(18)$$

$$a = 5$$

So the equation of the parabola is

$$y = 5x^2 + 15x - 350$$

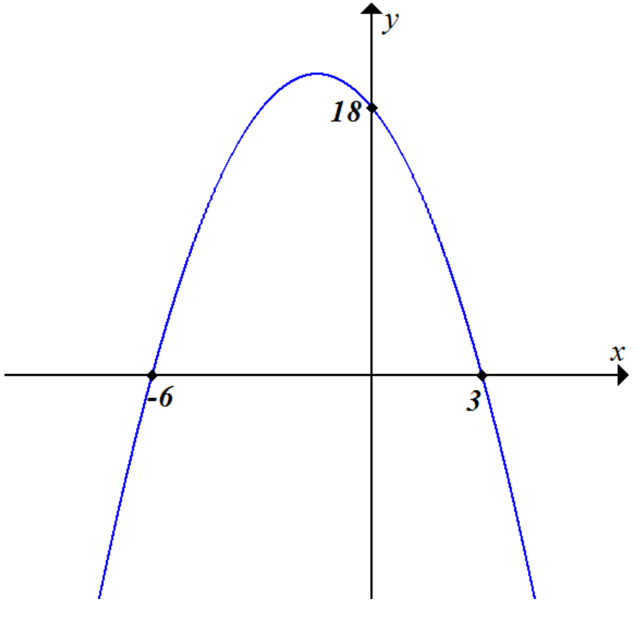
Question 7

c) We first factorise the given equation:

$$y = -(x^2 + 3x - 18)$$

$$y = -(x+6)(x-3)$$

So the graph passes through the points $(-6,0)$,$(3,0)$ and $(0,18)$ and it faces downwards:



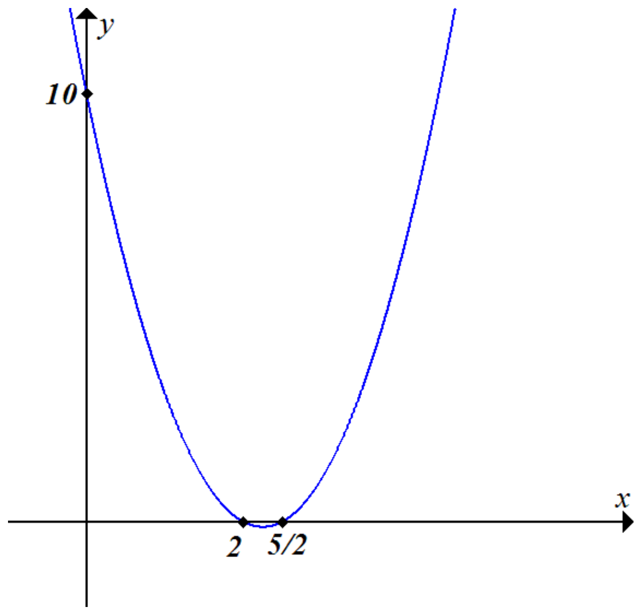
d) Again we start by factorising the given function:

$$y = 2x^2 - 9x + 10$$

$$y = 2\left(x^2 - \frac{9}{2}x + 5\right)$$

$$y = 2(x-2)\left(x - \frac{5}{2}\right)$$

So the graph passes through the points $(2{,}0)$, $\left(\frac{5}{2}, 0\right)$ and (0.10) and faces upwards:



Question 8

To answer the following questions, we will use the properties of parabolas that were found in Exercise 3C.

a) If the value of c is positive then the parabola crosses the y-axis at a positive value of y. So the following parabolas cross the y-axis at a positive y-value: A (expand the given equation to see the value of c), B, G, H.

b) The vertex is at the highest point of the graph if a is negative.

B, D, F (expand the given equation)

c) The vertex is to the left of the y-axis if a and b have the same sign:

F, G, H

d) A parabola passes through the origin if c is zero:

D

e) A parabola does not cross the x-axis at two separate points if its two factors are the same in tis factored form:

G

f) A parabola has the y-axis as its axis of symmetry if b is zero:

I

g) The axis of symmetry of a parabola lies exactly half-way between the points at which the parabola crosses the x-axis. If it only touches the x-axis rather than crosses the x-axis then the axis of symmetry is at the point at which the graph touches the x-axis:

B and E

h) For the vertex of a parabola to lie in the fourth quadrant, the axis of symmetry has to lie to the right of the y-axis and the graph has to face upwards and it has to cross the x-axis:

A, C, E

Question 9

a) Here both points at which the graph crosses the x-axis lie to the left of the y-axis. So the roots of the function are negative. The graph also points upward so a has to be positive and c has to be positive since the y-intercept is positive. Here we can choose a to be 1 since the graph has no given scale. In factor form a suggested equation is

$$y = (x+4)(x+2)$$

So in expanded form the equation is

$$y = x^2 + 6x + 8$$

e) This function is a cubic and it passes through the origin, so one of it is factors is x. The other roots of the function are negative. The constant in front of the factors has to be positive, since the graph is in the first quadrant when x is very large. A suggested equation is

$$y = x(x+2)(x+4)$$

MISCELLANEOUS EXERCISE 3

Question 1

$f(x) = 7x - 4$

a)

$$f(7) = 7(7) - 4 = 49 - 4 = 45$$

$$f\left(\frac{1}{2}\right) = 7\left(\frac{1}{2}\right) - 4 = \frac{7}{2} - 4 = -\frac{1}{2}$$

$$f(-5) = 7(-5) - 4 = -35 - 4 = -39$$

b)

$$f(x) = 10$$
$$7x - 4 = 10$$
$$7x = 14$$
$$x = 2$$

c)

$$f(x) = x$$
$$7x - 4 = x$$
$$6x = 4$$
$$x = \frac{4}{6} = \frac{2}{3}$$

d)

$$f(x) = 37$$
$$7x - 4 = 37$$
$$7x = 41$$

$$x = \frac{41}{7}$$

[Note: the answer for part (d) at the back of the text book is incorrect.]

Question 2

$$f(x) = f(4)$$

$$x^2 - 3x + 5 = 4^2 - 3(4) + 5$$

$$x^2 - 3x + 5 = 9$$

$$x^2 - 3x - 4 = 0$$

$$(x - 4)(x + 1) = 0$$

$$x = 4 \ \ or \ \ x = -1$$

Thus, at $x = 4$ and at $x = -1$, $f(x) = f(4)$.

Question 3

We are given that the curve passes between the point $(2,200)$ and $(2,2000)$, that is when $x = 2$, the value of y is greater 200 and less than 2000. So we can write

$$200 < y(2) < 2000$$

$$200 < 2^n < 2000$$

The given graph is not symmetrical about the y-axis but it is symmetrical about the origin. So n has to be odd. Now by inspection we see that $2^7 = 128$, $2^9 = 512$, and $2^{11} = 2048$. So clearly $n = 9$.

Question 4

$$y = x^2 - 7x + 5 \qquad (1)$$

$$y = 1 + 2x - x^2 \qquad (2)$$

(2) → (1):

$$-x^2 + 2x + 1 = x^2 - 7x + 5$$

$$2x^2 - 9x + 4 = 0$$

$$(2x - 1)(x - 4) = 0$$

$$x = \frac{1}{2} \ \ or \ \ x = 4$$

At $x = \frac{1}{2}$:

$$y = \left(\frac{1}{2}\right)^2 - 7\left(\frac{1}{2}\right) + 5$$

$$y = \frac{1}{4} - \frac{7}{2} + 5$$

$$y = \frac{7}{4}$$

At $x = 4$:

$$y = (4)^2 - 7(4) + 5$$

$$y = 16 - 28 + 5$$

$$y = -7$$

So the two curves intersect at $\left(\frac{1}{2}, \frac{7}{4}\right)$ and $(4, -7)$.

Question 7

$$y = (x-2)(x-4) \qquad (1)$$

$$y = x(2-x) \qquad (2)$$

Rearrange these equations:

$$y = x^2 - 6x + 8 \qquad (3)$$

$$y = 2x - x^2 \qquad (4)$$

(4) → (3):

$$2x - x^2 = x^2 - 6x + 8$$

$$2x^2 - 8x + 8 = 0$$

$$2(x^2 - 4x + 4) = 0$$

$$2(x-2)(x-2) = 0$$

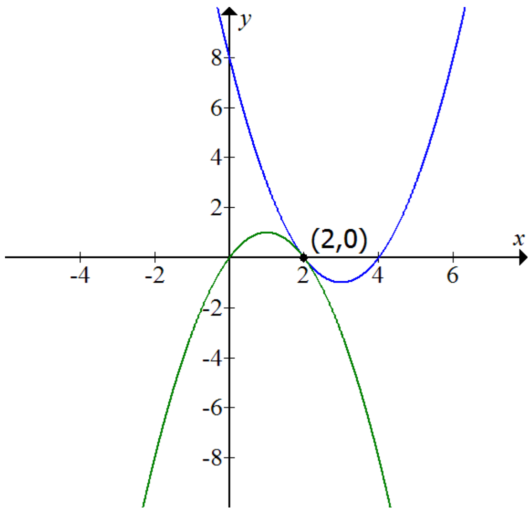
$$x = 2$$

At $x = 2$:

$$y = 2(2) - 2^2$$

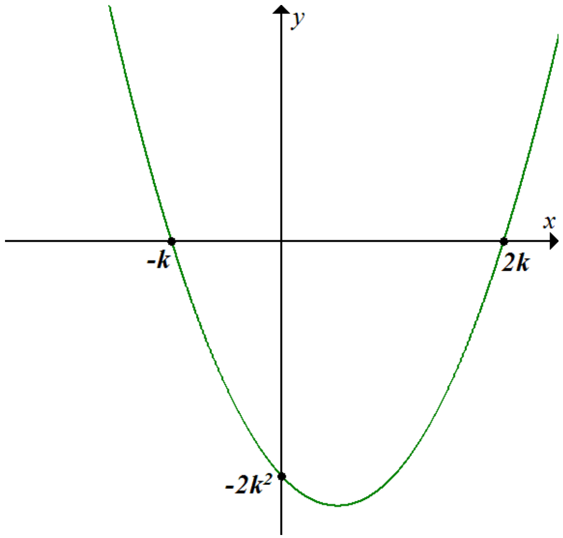
$$y = 4 - 4 = 0$$

So the two curves intersect at $(2,0)$. This is illustrated in the graph below.

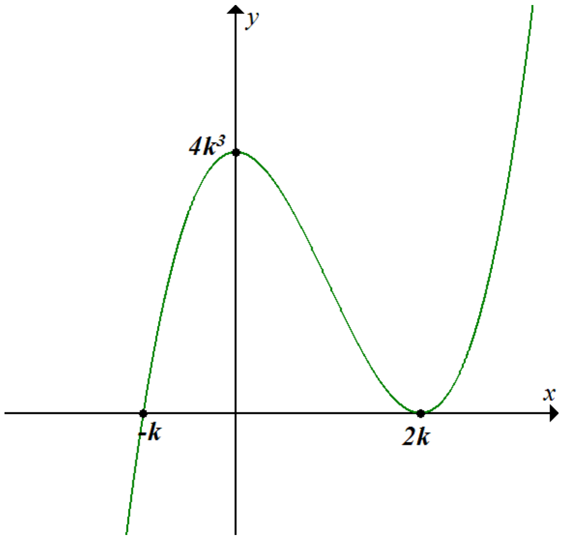


Question 8

a) The graph passes through the points $(-k, 0)$, $(2k, 0)$ and $(0, -2k^2)$. The axis of symmetry lies exactly between the first two points, where $x = -k + \frac{3}{2}k$. Now $\frac{3}{2}k > k$, so for any value of k, the axis of symmetry will lie to the right of the y-axis. The graph will also faces upwards:



d) Since the factor $(x - 2k)$ is squared the graph does not cross the x-axis at $(2k, 0)$ but it only touches the x-axis at that point. (The x-axis is tangent to the graph at this point.) It crosses the x-axis at $(-k, 0)$ and the y-axis at $(0, 4k^3)$. Since $k > 0$ the y-intercept is always positive. The constant before all the factors is 1 so the graph goes down to negative infinity for very small x and to positive infinity for very large x.



Question 9

We are given a function, $f(x) = ax^2 + bx + c$, and the value of the function at three points: $f(0) = 6$, $f(-1) = 15$, and $f(1) = 1$. So we can substitute the value of the function at these points into the expression for the function and solve the three resulting equations to find the values of $a, b,$ and c:

$$f(0) = a(0)^2 + b(x) + c = c = 6 \quad (1)$$

$$f(-1) = a(-1)^2 + b(-1) + c = 15 \quad (2)$$

$$f(1) = a(1)^2 + b(1) + c = 1 \quad (3)$$

Starting with eqn (2):

$$a - b + c = 15$$

But from eqn (1) we have $c = 6$ so we get

$$a - b + 6 = 15$$

$$a - b = 9 \qquad (4)$$

Now from eqns (1) and (3) we get

$$a + b + 6 = 1$$

$$a = -5 - b \qquad (5)$$

(5) → (4):

$$(-5 - b) - b = 9$$

$$-2b = 14$$

$$b = -7 \qquad (6)$$

(6) → (5):

$$a = -5 - (-7) = 2$$

So the function is $f(x) = 2x^2 - 7x + 6$.

Question 11

a) Notice that the given function is the sum of two parabolas. But, the sum of two parabolas is still a parabola, so the easiest way to graph this function is to expand and simplify the given expression and then to factorise the resulting expression:

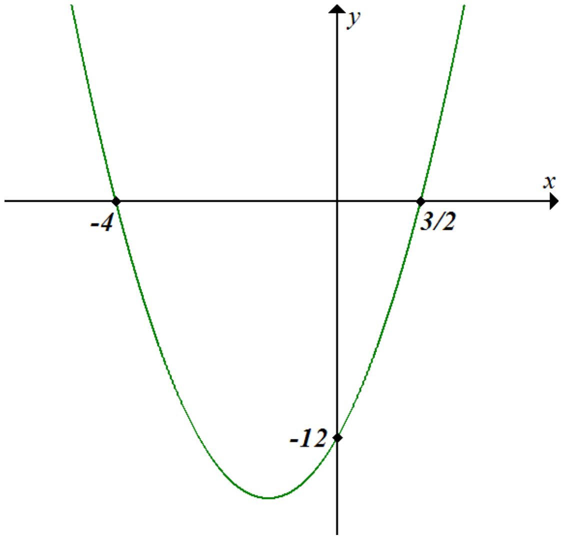
$$y = (x + 4)(x + 2) + (x + 4)(x - 5)$$

$$y = x^2 + 6x + 8 + x^2 - x - 20$$

$$y = 2x^2 + 5x - 12$$

$$y = (2x - 3)(x + 4)$$

Now it is easy to see that the graph intersects the x-axis at the points $\left(\frac{3}{2}, 0\right)$ and $(-4, 0)$. The graph intersects the y-axis at $(0, -12)$ and faces upwards.



Question 13

Again, we can find three equations which we can use to solve for a, b, and c:

At $x = -4$ we have that $y = 0$:

$$a(-4)^2 + b(-4) + c = 0$$

$$16a - 4b + c = 0 \qquad (1)$$

At $x = 9$, $y = 0$:

$$a(9)^2 + b(9) + c = 0$$

$$81a + 9b + c = 0 \qquad (2)$$

At $x = 1$, $y = 120$:

$$a(1)^2 + b(1) + c = 120$$

$$a + b + c = 120 \qquad (3)$$

Rearrange eqn (1):

$$c = 4b - 16a \qquad (4)$$

(4) → (2):

$$81a + 9b + 4b - 16a = 0$$

$$65a + 13b = 0$$

$$13(5a + b) = 0$$

$$5a + b = 0$$

$$b = -5a \qquad (5)$$

(5) → (4):

$$\begin{aligned} c &= 4(-5a) - 16a \\ &= -20a - 16a \\ &= -36a \qquad (6) \end{aligned}$$

(6) & (5) → (3):

$$a - 5a - 36a = 120$$

$$-40a = 120$$

$$a = -3 \qquad (7)$$

(7) → (5):

$$b = -5(-3) = 15$$

(7) → (6):

$$c = -36(-3) = 108$$

So the equation of the curve is $y = -3x^2 + 15x + 108$. So the curve crosses the x-axis at $(0,108)$.

Question 14

We can solve for a, b, and c using the same method as in question 13:

$$a(-1)^2 + b(-1) + c = 22$$

$$a - b + c = 22 \qquad (1)$$

$$a(1)^2 + b(1) + c = 8$$

$$a + b + c = 8 \qquad (2)$$

$$a(3)^2 + b(3) + c = 10$$

$$9a + 3b + c = 10 \qquad (3)$$

Rearrange eqn (1):

$$a = 22 + b - c \qquad (4)$$

(4) → (2):

$$22 + b - c + b + c = 8$$

$$2b = -14$$

$$b = -7 \qquad (5)$$

(5) → (4):

$$a = 22 - 7 - c$$

$$a = 15 - c \qquad (6)$$

(6) & (5) → (3):

$$9(15 - c) + 3(-7) + c = 10$$

$$135 - 9c - 21 + c = 10$$

$$-8c = -104$$

$$c = 13 \qquad (7)$$

(7) → (6):

$$a = 15 - 13 = 2$$

So the equation of the curve is $y = 2x^2 - 7x + 13$. The value of p is the value of the curve at $x = -2$:

$$y = 2(-2)^2 - 7(-2) + 13$$

$$y = 8 + 14 + 13$$

$$y = 35$$

So $p = 35$.

The possible values of q are those values of x such that $y = 17$:

$$17 = 2x^2 - 7x + 13$$

$$2x^2 - 7x - 4 = 0$$

$$(2x + 1)(x - 4) = 0$$

$$x = -\frac{1}{2} \ \ or \ \ x = 4$$

So the possible values of q are $-\frac{1}{2}$ and 4.

Question 17

Let us first we find the points of intersection of the first two curves:

$$y = x^2 + 3x + 14 \qquad (1)$$

$$y = x^2 + 2x + 11 \qquad (2)$$

(1) → (2):

$$x^2 + 3x + 14 = x^2 + 2x + 11$$

$$x + 3 = 0$$

$$x = -3$$

At $x = -3$:

$$y = (-3)^2 + 3(-3) + 14$$

$$y = 9 - 9 + 14$$

$$y = 14$$

So the first two curves intersect at only one point, $(-3,14)$. Thus the third curve also has to pass through this point. This enables us to find the value of p:

$$14 = p(-3)^2 + p(-3) + p$$

$$14 = 9p - 3p + p$$

$$14 = 7p$$

$$p = 2$$

So the equation of the third curve is $y = 2x^2 + 2x + 2$.

Question 23

$$y = 2x^2 - 7x + 14 \qquad (1)$$

$$y = 2 + 5x - x^2 \qquad (2)$$

(1) → (2):

$$2x^2 - 7x + 14 = 2 + 5x - x^2$$

$$3x^2 - 12x + 12 = 0$$

$$3(x^2 - 4x + 4) = 0$$

$$3(x - 2)(x - 2) = 0$$

$$x = 2$$

At $x = 2$:

$$y = 2 + 5(2) - 2^2$$

$$y = 2 + 10 - 4$$

$$y = 8$$

So the two curves meet at only one point, $(2,8)$.

a) Here the first curve is shifted down by two units while the second curve remains the same. Now vertex of the second curve is at the maximum point of the graph. So shifting the first curve down by two units means that the two curves will now intersect at two points.

b) Here the first curve remains the same while the second curve is shifted down by one unit. Since the vertex of the first curve is at the minimum point of the graph, shifting the second curve down results in the two curves no longer intersecting.

c) Now both curves are shifted up by twenty units. Since there is no relative change, the curves will still intersect at only one point.

Question 24

a) If $x > 0$, then we know that $|x| > 0$ and that $\frac{1}{x} > 0$ (the inequality sign is not reversed here since we are dealing with zero and 1 divided by a positive number is still positive). Now x and $|x|$ have the same magnitude so we can say that

$$\frac{|x|}{x} = 1$$

b) If $x < 0$, we know that $|x| > 0$ and that $\frac{1}{x} < 0$. Now a negative number times a positive number is a negative number, so we can say that

$$\frac{|x|}{x} = -1$$